p-primary torsion 0-cycle

§ Intro.

X : smooth proj / k

$$CH^r(X) = \bigoplus_{\operatorname{codim} Z = r} \mathbb{Z}\cdot Z \Big/ \text{rat. eq.}$$

予想. $[k:\mathbb{Q}] < \infty$ $CH^r(X)$ は f.g. $\mathbb{Z}$-mod. □

(特に torsion part は有限)

・$r=1$ OK.

$$0 \to \underbrace{\text{Picard}(k)}_{\text{f.g. } \mathbb{Z}\text{-mod.}} \to CH^1(X) \to \underbrace{NS(X)}_{\text{f.g.}} \to 0$$

(Mordell-Weil)

・$r \geq 2$ 全く未解決

以下 $(k:\mathbb{Q}_p) < \infty$ とする.

Thm. (Mattuck, 1955)

A : abel var. / k $\quad g = \dim A$

$\mathcal{O}_k$: 整数環

$\Longrightarrow A(k) \cong \mathcal{O}_k^{\oplus g} + (\text{fin. gp.})$ □

問 $(k:\mathbb{Q}_p) < \infty$ $CH^r(X)_{tor}$ 有限 ?

反例あり

2007 Rosenschon-Srinivas $\exists X$: 4-fold / k

$CH^3(X)_{tor} = \infty$

2008 S. Saito - Asakura $\exists X \subset \mathbb{P}^3$: degree 5以上の曲面で good reduction Y をもつ

$r = \operatorname{rank} NS(Y)$

$$\ell \neq p \quad CH_0(X)[\ell^\infty] = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r-1} + (\text{fin.})$$

$\quad \| \quad CH^2(X)$

定理 $\exists X \subset \mathbb{P}^3_{\mathbb{Q}_p}$: 5次曲面 $CH_0(X)[p^\infty] = \infty$ ∎

§ $CH^2(X,1)$ と $CH^2(X)_{tor}$ の関係

$$K_2^M(k(X)) := k(X)^* \otimes k(X)^* \big/ \langle f \otimes (1-f) \rangle$$

Milnor K_2 $\quad f \otimes g$ を $\{f,g\}$ とかく

$$d_2 : K_2^M(k(X)) \longrightarrow \bigoplus_{\operatorname{codim} D = 1} k(D)^*$$

$$\{f,g\} \longmapsto \sum_D (-1)^{\operatorname{ord}_D(f)\operatorname{ord}_D(g)} \frac{f^{\operatorname{ord}_D(g)}}{g^{\operatorname{ord}_D(f)}}\Big|_D$$

$$d_1 : \bigoplus_{\operatorname{codim} D = 1} k(D)^\times \longrightarrow \bigoplus_{\operatorname{codim} Z = 2} \mathbb{Z}$$

$$\in$$

$$f_D \otimes D \longmapsto j_* \operatorname{div}_{\tilde{D}}(f_D)$$

$\tilde{D} \xrightarrow{j} D$: normalization

Fact $d_1 d_2 = 0$

定義 $CH^2(X,1) := \operatorname{Ker} d_1 / \operatorname{Im} d_2$

$$CH^2(X,1,\mathbb{Z}/n) := \operatorname{Ker}(d_1 \otimes \mathbb{Z}/n) / \operatorname{Im}(d_2 \otimes \mathbb{Z}/n)$$

Rem. $CH^2(X) = \operatorname{Coker} d_1$

$$0 \longrightarrow CH^2(X,1)/n \longrightarrow CH^2(X,1,\mathbb{Z}/n) \longrightarrow CH^2(X)[n] \longrightarrow 0 \quad (e)$$

$$\underset{\text{Bloch-Ogus theory}}{\Longrightarrow} CH^3(X,1,\mathbb{Z}/n) \underset{\text{レギュレーター写像}}{\xrightarrow{\cong}} NH^3_{\text{ét}}(X, \mathbb{Z}/n(2))$$

$$NH^3_{\text{ét}} := \operatorname{Ker}(H^3(X) \longrightarrow H^3(k(X)))$$

$(k : \mathbb{Q}_p) < \infty$ とする

$$CH^2(X)[p^\infty] \cong NH^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) / CH^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

$$\underset{(k:\mathbb{Q}_p)<\infty \text{ より}}{\cong} (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus m} + (\text{fin.})$$

m を corank とよぶ

§証明

$$X : \underset{5次}{F(x,y,z)} + w\, \underset{4次}{G(x,y,z,w)} = 0 \quad \text{in } \mathbb{P}^3(x,y,z,w)$$

(1) F, G は $\mathbb{Z}_p$ 係数

$Y = X \bmod p$ が smooth

$(X \times_{\mathbb{Z}_p} \mathbb{F}_p)$

(2) $G = \sum_I d_I x^I$

$$F = x^2(x-z)^2(a_0 x + a_1 y + a_2 z) + y^2(y-z)^2(b_0 x + b_1 y + b_2 z)$$
$$+ xy(x-z)(y-z)(c_0 x + c_1 y + c_2 z)$$

となっていて $\{d_I, a_i, b_j, c_k\}$ $\mathbb{Q}$上代数独立

(Rem. $F=0$ は node を4つもつ既約曲線)

(3) $F \bmod p$ がある条件をみたす

corank $CH_0(X)[p^\infty] \geq 1$ を示したい.

① $CH^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p = (\underset{\{c \otimes H\},\ c:\ \text{const.}}{\mathbb{Q}_p^\times \otimes \underset{超平面切断}{H}}) \otimes (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}$

② corank $NH^3 \geq 3$

① (略) [AS] と同じ proof

② $CH^2(X,1,\mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow[\sim]{B.O.} NH^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) \supset \mathbb{Q}_p^\times \otimes H$

$\downarrow \partial_p$ = boundary

$\mathrm{Pic}(Y) \otimes \mathbb{Q}_p/\mathbb{Z}_p$

示すこと

$\mathrm{corank}(\mathrm{Im}\,\partial_p) \geq 2$

Rem. ℓ進の場合

$CH^2(X,1,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\partial_p} \mathrm{Pic}(Y) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$

が mod. finite で同型

(佐藤、斎藤秀司 Ann. of Math を使って証明)

$C : F = w = 0$ node を A_1, A_2, A_3, A_4

$\tilde{C} \xrightarrow{j} C$ normalization $j^{-1}(A_i) = \{P_i, Q_i\}$

$g = 2$

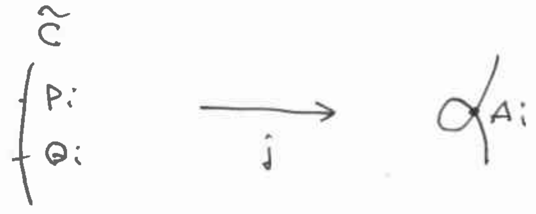
$P_i - Q_i \in J(\tilde{C})(\mathbb{Q}_p) = \mathbb{Z}_p^{\oplus 2} + (\mathrm{fin.})$

${}^{\exists}\sum r_i(P_i - Q_i) = 0$ i.e. ${}^{\exists} f_n \in \mathbb{Q}_p(C)^\times$

$r_i \in \mathbb{Z}_p$ $\qquad \mathrm{div}_{\tilde{C}}(f_n) \equiv \sum r_i(P_i - Q_i) \mod p^n$

$\xi_n := f_n \otimes C \in CH^2(X,1,\mathbb{Z}/p^n)$

$\Longrightarrow$ ② を示すときに使う元



$j_*(\mathrm{div}\, f_n) = \sum (A_i - A_i) \mod p^n = 0$

Lem. $\mathrm{ord}_p(a) > 0$, $a \in \mathbb{Z}_p$

$E : y^2 = (x^2 + a) f(x)^2$

$f(x) = \prod_i (x - \alpha_i) \qquad \alpha_i \in \mathbb{Z}_p$

$\alpha_i \not\equiv \alpha_j \mod p \qquad \alpha_i \not\equiv 0 \mod p$ $\qquad \square$

$E \mod p = E_1 + E_2$

$$g = \frac{y - (x + \sqrt{\alpha_i^2 + p} - \alpha_i) f(x)}{y - (x - \sqrt{\alpha_i^2 + p} - \alpha_i) f(x)} \Longrightarrow \underbrace{\mathrm{div}_{E_{\mathbb{Z}_p}}(f)}_{\substack{\parallel \\ \partial_p(g \otimes E)}} = E_1$$