# 局所体上の楕円曲線のKummer写像の像について

§1. Main Theorem

§2. Motivation of main theorem

## §1

$p$: prime

$K/\mathbb{Q}_p$: finite extension, $\mathcal{O}_K$: integer ring of $K$

$\mathfrak{m}_K \subset \mathcal{O}_K$: maximal ideal of $K$, $F_K := \mathcal{O}_K/\mathfrak{m}_K$

$e$: absolute ramification index of $K$

$c_k := ke + \frac{e}{p-1}$ $(k = 1, \dots, n)$ $n \in \mathbb{N}_{\geq 1}$

$s \geq 1$ に対して

$$U_K^{(s)} := 1 + \mathfrak{m}_K^s$$

$$U_{K,n}^{(s)} := U_K^{(s)} / (U_K^{(s)} \cap (K^\times)^{p^n})$$

$$Gr_{K,n}^{(s)} := U_{K,n}^{(s)} / U_{K,n}^{(s+1)}$$

**Main theorem** $\mu_{p^n} \subset K^\times$ と仮定

(1) $1 \leq s < c_1 \Rightarrow Gr_{K,n}^{(s)} \cong \begin{cases} F_K^+ & p^n \nmid s \\ 1 & p^n \mid s \end{cases}$

(2) $\sqrt[p-1]{-\binom{p^n}{p^{n-1}}} \in K$ と仮定

$s = c_1 \Rightarrow Gr_{K,n}^{(s)} \cong \mathbb{Z}/p\mathbb{Z}$

(3) $c_k < s < c_{k+1}$, $p^{n-k} \mid s$ $(k = 1, \dots, n-1)$

$\Rightarrow Gr_{K,n}^{(s)} = 1$

(4) $\sqrt[p-1]{-\binom{p^n}{p^{n-k}}}$, $\sqrt[p-1]{-\binom{p^n}{p^{n-k+1}}} \in K$ と仮定

$s = c_k$ $(k = 2, \dots, n) \Rightarrow Gr_{K,n}^{(s)} \cong \mathbb{Z}/p\mathbb{Z}$ or $1$

(5) $s > c_n \Rightarrow Gr_{K,n}^{(s)} = 1$ (竹本氏)

特に $n = 2$ の場合はさらに次が成立

(3)' $c_1 < s < c_2$, $p \nmid s \Rightarrow Gr_{K,2}^{(s)} \cong F_K^+$

(4)' $s = c_2 \Rightarrow Gr_{K,2}^{(s)} \cong \mathbb{Z}/p\mathbb{Z}$ □

**Theorem** $K/\mathbb{F}_p((T))$: 有限次拡大

$s \geq 1$ に対して $Gr_{K,n}^{(s)} \cong \begin{cases} F_K^+ & p^n \nmid s \\ 1 & p^n \mid s \end{cases}$ □

## §2

$E, E'/K$: 楕円曲線

$\nu: E \to E'$: $p^n$-isogeny (i.e. $\# \ker \nu = p^n$)

$0 \to \ker \nu \to E \to E' \to 0$ (完全)

$\rightsquigarrow 0 \to E'(K)/\nu E(K) \xrightarrow{\delta_n} H^1(K, \ker \nu) \to H^1(K, E)$

$\delta_n$: 局所Kummer写像

**Question** $\mathrm{Im}\,\delta_n$ の構造はどうなるか？

- ordinary case と multiplicative case はすでに知られている。
  (竹本氏)　　(山崎先生)
- supersingular case は $n=1$ のときのみ知られている。(川内氏)

$\boxed{n=1}$ $i \geq 1$ $E_i(K) := \{(x,y) \in E(K) \mid v(x) \leq -2i,\ v(y) \leq -3i\}$ 　($v$: $K$の正規付値)

$\pi : E(K) \longrightarrow \tilde{E}(F_K)$ : 還元写像

$E_0(K) := \pi^{-1}(\tilde{E}_{ns}(F_K))$

$D_K := E'(K)/\nu E(K)$, $D_K^i := E_i'(K)/\nu E(K) \cap E_i'(K)$

$\hat{E}$, $\hat{E}'$ : 楕円曲線に付随する形式群 / $O_K$

$\hat{\nu} : \hat{E} \longrightarrow \hat{E}'$ : $p^n$-isogeny

$\hat{E}'^i := \hat{E}(m_K^i)$ 　$(i \geq 1)$

$\mathcal{D}_K^i := \hat{E}'^i / \nu\hat{E}' \cap \hat{E}'^i$ 　$(i \geq 1)$

$\mathrm{Ker}\,\nu \subset E(K)$ と仮定. Weil pairing より $\mu_p \subset K^\times$

$$1 \leq i < pt \Rightarrow \mathcal{D}_K^i/\mathcal{D}_K^{i+1} \cong \begin{cases} F_K^+ & p \nmid i \\ 1 & p \mid i \end{cases} \cong Gr_{K,1}^{(i+(e_0-t)p)}$$

$$i = pt \Rightarrow \mathcal{D}_K^i/\mathcal{D}_K^{i+1} \cong \mathbb{Z}/p\mathbb{Z} \cong Gr_{K,1}^{(i+(e_0-t)p)}$$

$$i > pt \Rightarrow \mathcal{D}_K^i/\mathcal{D}_K^{i+1} = 1 = Gr_{K,1}^{(i+(e_0-t)p)}$$

$$\begin{array}{ccc} \#(\mathcal{D}_K^1/\mathcal{D}_K^2)\cdot\#(\mathcal{D}_K^2/\mathcal{D}_K^3)\cdots & = & \#Gr_{K,1}^{(1+(e_0-t)p)}\cdot\#Gr_{K,1}^{(2+(e_0-t)p)}\cdots \\ \parallel & & \parallel \\ \#\mathcal{D}_K^1 & & \#U_{K,1}^{(1+(e_0-t)p)} \end{array}$$

$$\therefore\ \delta(\mathcal{D}_K^1) = U_{K,1}^{(1+(e_0-t)p)}$$

$$\begin{tikzcd}
D_K \arrow[rr, "\delta_1"] & & H^1(K,\mathrm{Ker}\,\nu) \\
D_K^0 \arrow[u, phantom, "\cup"] \arrow[u, phantom, "\text{good reduction}\ \parallel"'] & \circlearrowright & K^\times/(K^\times)^p \arrow[u, phantom, "\cong"] \\
& & U_{K,1}^{(0)} \arrow[u, phantom, "\cup"] \\
D_K^1 \arrow[uu, phantom, "\cup"] \arrow[uu, phantom, "\text{supersingular}\ \parallel"'] & \mathcal{D}_K^1 \arrow[l, phantom, "\cong"] \arrow[uur, "\delta"] \arrow[ddr, bend right] & U_{K,1}^{(1)} \arrow[u, phantom, "\cup"] \\
D_K^2 \arrow[u, phantom, "\cup"] & \mathcal{D}_K^2 \arrow[u, phantom, "\cup"] & \vdots \arrow[u, phantom, "\cup"] \\
\vdots \arrow[u, phantom, "\cup"] & \vdots \arrow[u, phantom, "\cup"] & U_{K,1}^{(1+(e_0-t)p)} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\left(e_0 = \frac{e}{p-1},\ 1 \leq t < \frac{p}{p-1}\right)$

## §3 $n=2$ のときの証明をする

(1) $p^2 \mid s$ のとき $Gr_{K,2}^{(s)} \cong U_K^{(s)}/U_K^{(s+1)}(U_K^{(s)} \cap (K^\times)^{p^2}) = 1$

$p^2 \nmid s$ のとき $F_K^+ \cong U_K^{(s)}/U_K^{(s+1)} \twoheadrightarrow U_K^{(s)}/U_K^{(s+1)}(U_K^{(s)} \cap (K^\times)^{p^2}) \cong Gr_{K,2}^{(s)}$

(2) $r = \dfrac{e}{p(p-1)}$

$$\begin{array}{ccccccccc} 1 \longrightarrow & \mathrm{Ker}\,p^2 & \longrightarrow & U_K^{(r)}/U_K^{(r+1)} & \xrightarrow{p^2\text{乗}} & U_K^{(c_1)}/U_K^{(c_1+1)} & \xrightarrow{\varphi} & Gr_{K,2}^{(s)} & \longrightarrow 1 \ (\text{完全}) \\ & \wr\wr & & \wr\wr & & \wr\wr & & & \\ & \mathbb{Z}/p\mathbb{Z} & & F_K^+ & & F_K^+ & & & \end{array}$$

(3) <u>Proposition</u> $s > c_1$ のとき 次は完全

$$1 \longrightarrow U_{K,1}^{(s-e)} \xrightarrow{\text{降}} U_{K,2}^{(s)} \longrightarrow U_{K,1}^{(s)} \longrightarrow 1$$ □

Prop. より 完全列

$$1 \longrightarrow Gr_{K,1}^{(s-e)} \longrightarrow Gr_{K,2}^{(s)} \longrightarrow \underset{\substack{\shortparallel \\ 1}}{Gr_{K,1}^{(s)}} \longrightarrow 1$$

が得られる.

$c_1 < s < c_2$ なので

$$\frac{e}{p-1} < s-e < \frac{pe}{p-1} \Rightarrow Gr_{K,1}^{(s-e)} \cong \begin{cases} F_K^{+} & p \nmid s \\ 1 & p \mid s \end{cases}$$

(4), (5) : (3) と同様.