On the governing fields for tame kernels of quadratic fields

Iwao KIMURA

(University of Toyama)

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Aim of this talk

- What is the tame kernel of number fields ?
- What is governing fields ?
- Why do governing fields matter?

Plan

1. Tame kernel of number fields
2. The governing fields for 2-power ranks of ideal class groups of quadratic fields
3. Known facts about the governing fields for 2-power ranks of ideal class groups of quadratic fields (our model case)
4. some known results for 2-power ranks of tame kernels associated with quadratic fields
5. Hurrelbrink-Kolster's 4-rank formulae [HK98]
6. toward a governing field for 4-rank of tame kernels associated with quadratic fields

Milnor's K_2 of a number field

F: a number field of finite degree over the rationals $\mathbf{Q}$, the second Milnor K-group $K_2(F)$ is defined by

$$K_2(F) := F^\times \otimes F^\times / \langle x \otimes (1-x) |\ x \in F^\times,\ x(1-x) \neq 0 \rangle.$$

The class represented by $a \otimes b \in F^\times \otimes F^\times$ is denoted by $\{a, b\} \in K_2(F)$.

Milnor's K_2 of a number field (cont'd)

S: a finite set of finite places of F, $O_S(F)$: the ring of S-integers of F,

$O_S^\times(F)$: the group of S-units of F,

$$K_2^S(F) := \{\{a,b\} \in K_2(F) |\ a, b \in O_S^\times(F)\}.$$

Note that $K_2^S(F)$ is finitely generated (since $O_S^\times$ is so).

S_m: the first m finite places of F with respect to the norm $N(v)$ of v, then it holds that

$$K_2(F) = \varinjlim_m K_2^{S_m}(F).$$

Tame symbol at a finite place v

Let v be a finite place of F, $k(v)$ be the residue field at v, then the **tame symbol** ∂_v at v is defined by

$$\partial_v : K_2(F) \to k(v)^\times, \quad \{a, b\} \mapsto (-1)^{\alpha\beta} \frac{a^\beta}{b^\alpha} \pmod{v},$$

where $\alpha = \mathrm{ord}_v(a)$, $\beta = \mathrm{ord}_v(b)$, $\mathrm{ord}_v(\cdot)$ is the additive normalized valuation at v.

Tame kernel of number fields

We define the **tame kernel** $K_2(O_F)$ of a number field F (whose ring of integers O_F) to be

$$K_2(O_F) := \bigcap_{v\text{: fin. places}} \ker(\partial_v : K_2(F) \to k(v)^\times).$$

Fact. The tame kernel of number field F is coincide with the second algebraic K-group of O_F.

Finiteness of tame kernels

Fact (Garland [Gar71]). $\exists S$: a finite set of finite places such that

$$K_2(O_F) \subset K_2^S(F).$$

Thus $K_2(O_F)$ is finitely generated. It is known that the groups is torsion. It follows from these fact that $\mathbf{K_2(O_F)}$ **is a finite abelian group**.

Computation of tame kernels

Tame kernel $K_2(O_F)$ of a number field F is computable in theory:

- its order
- its structure

cf. a practical algorithm is given by Belabas-Gangl [BG04].

If F is a real abelian field, the order of $K_2(O_F)$ is given by the formula (Birch-Tate conjecture, proved by Mazur-Wiles, Kolster)

$$\#K_2(O_F) = (-1)^{[F:\mathbf{Q}]} w_2(F) \zeta_F(-1),$$

where $w_2(F) := \max\{n | \exp(\mathrm{Gal}(F(\zeta_n)/F)) \leq 2\}$.

Distribution of (odd parts of) tame kernels of quadratic fields

$F = \mathbf{Q}(\sqrt{D})$: a quadratic field of the discriminant D,

O_D: its ring of integers,

p: an odd prime (fix),

Problem: For a positive real number X, estimate the number

$$\#\{0 < |D| < X \mid p \nmid \#K_2(O_D)\}$$

in terms of X.

If $D > 0$, one can obtain some estimate by using Birch-Tate conjecture ([Kim07]).

Distribution of (odd parts of) tame kernels of quadratic fields (cont'd)

With the same notations,

Problem: For a positive real number X, estimate the number

$$\#\{0 < |D| < X \,|\, p \mid \#K_2(O_D)\}$$

in terms of X.

(For $p = 3$, if $d > 0$ and $d \equiv 6 \pmod 9$ then $3 \mid \#K_2(O_d)$, by J. Browkin [Bro00].

For $p = 5$, if $d > 0$, $5 \mid h(\mathbf{Q}(\sqrt{5d}))$ then $5 \mid \#K_2(O_d)$ by [Bro92][a].)

[a] Just after my talk, Prof. Y. Kishi kindly noticed me that one can deduce, from Ichimura [Ich03], there are infitely many real quadratic fields whose class numbers and discriminants both divisible by 5. Thus we see $\exists^\infty D > 0$ such that $5 \mid \#K_2(O_D)$. This has been shown already by Kimura [Kim06].

2-power ranks for finite abelian groups

Notation. G: a finite abelian group,

2^i-rank $e_i(G)$ of G $(i = 1, 2, \ldots)$ are defined by

$$e_i(G) = \dim_{\mathbf{Z}/2\mathbf{Z}}(G^{2^{i-1}}/G^{2^i}).$$

Distribution of (2-parts of) tame kernels of quadratic fields

Today's theme: We want to know the following density of prime numbers q:

D: A square free integer (fix).

e: A natural number (fix).

$\mathcal{T}$: A finite abelian 2-group of exponent dividing 2^e (fix).

$$\frac{\#\{q \mid K_2(O_{Dq})/K_2(O_{Dq})^{2^e} \cong \mathcal{T}\}}{\#\{\text{all primes}\}} = ?,$$

where O_{Dq} is the ring of integers of $\mathbf{Q}(\sqrt{Dq})$.

Model Case: 2-part of ideal class groups

D: A square free integer (fix).

e: A natural number (fix).

$\mathcal{T}$: A finite abelian 2-group of exponent dividing 2^e (fix).

$$\frac{\#\{q|\ \mathrm{Cl}(O_{Dq})/\mathrm{Cl}(O_{Dq})^{2^e} \cong \mathcal{T}\}}{\#\{\text{all primes}\}} = ?,$$

where $\mathrm{Cl}(O_{Dq})$ is the ideal class group of $\mathbf{Q}(\sqrt{Dq})$.

In some cases, **the RHS is known!**

Model Case: Governing field for 2-part of ideal class groups

Fact. (Stevenhagen [Ste89], Morton [Mor82],...) For a square free integer D (with some assumptions), there is a Galois extension $\Sigma(D)/\mathbf{Q}$ such that the following equivalence holds: for a triple of integers ρ, s and r ($0 \leq \rho \leq s \leq r$),

$$\mathrm{Cl}(O_{Dq})/\mathrm{Cl}(O_{Dq})^8 \cong (\mathbf{Z}/2\mathbf{Z})^{r-s} \oplus (\mathbf{Z}/4\mathbf{Z})^{\rho} \oplus (\mathbf{Z}/8\mathbf{Z})^{s-\rho}$$

$$\Longleftrightarrow$$

$$\left[\frac{\Sigma(D)/\mathbf{Q}}{q}\right] \subset \text{Conjugacy classes depending on } \rho,\ s \text{ and } r.$$

Chebotarev density theorem provides the density of such q.

Governing field for ideal class group (Cohn-Lagarias) [CL83]

This kind of phenomenon was first suggested by Cohn-Lagarias 1983.

Governing field for ideal class group (Morton) [Mor82]

Suppose $D = p_1 \ldots p_r$, $p_i \equiv 1 \pmod 4$, $\left(\frac{p_i}{p_j}\right) = 1$ for $i \neq j$, $q \equiv 3 \pmod 4$,

then, $\exists \Sigma(-D)$ such that $\left[\frac{\Sigma(-D)/\mathbf{Q}}{q}\right]$ determines $\mathrm{Cl}(-Dq)/\mathrm{Cl}(-Dq)^8$.

Further, Morton shows that, in this case, $[\Sigma(-D) : \mathbf{Q}] = 2^{\binom{r}{2}+2r}$ and gave explicit density.

(*cf.* Hokuriku Number Theory Workshop 2007.)

Governing field for ideal class group (Stevenhagen) [Ste89]

For any $D \in \mathbf{Z}$, $D \not\equiv 2 \pmod 4$,

$$K_D := \mathbf{Q}(\sqrt{p^*}; p^* \mid D),$$

where p^* is a prime fundamental discriminant.

$\Omega(D) :=$ the maximal abelian extension of K_D unramified outside $2D\infty$ and of exponent 2 over K_D.

Then, $\mathrm{Cl}(Dq)/\mathrm{Cl}(Dq)^8$ is determined by $\left[\frac{\Omega(D)/\mathbf{Q}}{q}\right]$.

(the most general up to now, but less explicit).

Morton's strategy

2-rank of ideal class groups of quadratic fields $\mathbf{Q}(\sqrt{Dq})$...well known,

4-rank and 8-rank are described by certain square matrix over $\mathbf{Z}/2\mathbf{Z}$.

Its entries are of the form

$$\left(\frac{N\mathfrak{p}_i, -Dq}{p_j}\right)',$$

where $'$ means that $1' = 0 \pmod 2$, $-1' = 1 \pmod 2$.

Strategy: decompose the matrix into the part depends only on D and depends on q.

4-rank formula of K_2

Hurrelbrink and Kolster [HK98, lemma 5.1].

For an imaginary quadratic field $\mathbf{Q}(\sqrt{d})$, $d < 0$,

$$e_2(K_2(O_d)) = \#\{p > 2;\ p \mid d,\} - \mathrm{rank}_{\mathbf{Z}/2\mathbf{Z}}(M(d)),$$

where $M(d)$ is the matrix of the form...

4-rank formula of K_2 (cont'd)

$$M(d) = \begin{pmatrix} (-d,p_1)_2 & (-d,p_1)_{p_1} & \cdots & (-d,p_1)_{p_t} \\ (-d,p_2)_2 & (-d,p_2)_{p_1} & \cdots & (-d,p_2)_{p_t} \\ \vdots & \vdots & \vdots & \vdots \\ (-d,p_{t-1})_2 & (-d,p_{t-1})_{p_1} & \cdots & (-d,p_{t-1})_{p_t} \\ (-d,v)_2 & (-d,v)_{p_1} & \cdots & (-d,v)_{p_t} \\ (-d,-1)_2 & (-d,-1)_{p_1} & \cdots & (-d,-1)_{p_t} \end{pmatrix}',$$

$v = 2$ if $2 \notin N(\mathbf{Q}(\sqrt{d})^\times)$, $v = u + w$ if $2 \in N(\mathbf{Q}(\sqrt{d})^\times)$ (in this case, $d \in N(\mathbf{Q}(\sqrt{2})^\times)$, so $d = u^2 - 2w^2$).

(Note that trailing $'$, this is a matrix over $\mathbf{Z}/2\mathbf{Z}$).

4-rank of K_2 for certain quadratic fields

Conner-Hurrelbrink [CH89] determined 4-ranks of K_2 for the following cases:

$$\begin{aligned} d &= pl, \quad 4-\text{rank} = 1 \text{ or } 2, \\ d &= 2pl, \quad 4-\text{rank} = 1 \text{ or } 2, \\ d &= -pl, \quad 4-\text{rank} = 0 \text{ or } 1, \\ d &= -2pl, \quad 4-\text{rank} = 0 \text{ or } 1. \end{aligned}$$

Method: Hurrelbrink-Kolster's 4-rank formula, relation between the rank of the matrix $M(dl)$ and splitting of l in certain number field, and representation of power of l by positive definite binary quadratic forms.

Osburn's computation of 4-rank densities [Osb02]

R. Osburn computed the 4-rank densities for $D = pl,\ 2pl,\ -pl,\ -2pl$.

$$\mathcal{L} = \left\{ l \in \mathbf{Z} |\ l \text{ is prime },\ l \equiv 1 \pmod 8, \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1 \right\}$$

Theorem (Osburn) For the fields $\mathbf{Q}(\sqrt{pl})$, $\mathbf{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density 1/2 in $\mathcal{L}$.

For the fields $\mathbf{Q}(\sqrt{-pl})$, $\mathbf{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density 1/2 in $\mathcal{L}$.

Method: a construnction of a **governing field** (no reference to this term, though).

4-rank formula revisited

The formula is of the form

$$e_2(K_2(O_{Dq})) = t - \mathrm{rank}_{\mathbf{Z}/2\mathbf{Z}} M(Dq).$$

If one can state the condition "If q is decomposed in certain way in a certain number field, then the rank of $M(Dq)$ is the same for those q", then the 4-rank is the same for those q.

(This gives an estimate of density of q from below.)

4-rank formula revisited (cont'd)

On the other hand, if one wants to compute the density of q which satisfies $e_2(K_2(O_{Dq})) = e$ (e given), one must enumerate possible $M(Dq)$.

(As Morton did in the ideal class groups case).

Conclusion

- Governing field is interesting notion (there also is a notion "Chebotarev set").
- Construction of a governing field for $K_2(O_{Dq})$ has established only for a few case (the case D having a few prime factors).
- 8-rank of $K_2(O_{Dq})$?...seems difficult. *cf.* Vazzana [Vaz99].

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