# Mahler Measure and Weber's Problem in the Cyclotomic $\mathbb{Z}_3$-extension of $\mathbb{Q}$

## §1

$K/\mathbb{Q}$ : fin. $h(K)$ : class number of $K$

$\mathbb{B}_n = \mathbb{Q}(2\cos\frac{2\pi}{3^{n+1}})$ , $h_n = h(\mathbb{B}_n)$

**Th. 1**

$\ell$ : prime $\not\equiv \pm 1 \ (27) \Rightarrow \ell \nmid h_n \ (\forall n)$ $\square$

## §2 Motivation

**Conj.** $\exists \infty \ K/\mathbb{Q}$ : fin. s.t. $h(K) = 1$ $\square$

$\leadsto$ " cyclo. $\mathbb{Z}_p$-ext. $/\mathbb{Q}$ "

$p$ : prime number

$\mu_m$ : gp. of all $m$-th roots of $1$

$\mathbb{Q}(\mu_{p^\infty}) := \bigcup_{n \geq 1} \mathbb{Q}(\mu_{p^n})$

$\exists 1 \ \mathbb{B}_{p,n}$ : real $\subseteq \mathbb{Q}(\mu_{p^\infty})$ : $\mathrm{Gal}(\mathbb{B}_{p,n}/\mathbb{Q}) \cong \mathbb{Z}/p^n\mathbb{Z}$

**Rmk.**

$$\begin{tikzcd} & \mathbb{Q}(\mu_{3^{n+1}}) \arrow[d, dash, "2"] \\ \mathbb{Q}(\mu_3) \arrow[ur, dash, "3^n"] \arrow[d, dash, "2"'] & \mathbb{B}_{3,n} \\ \mathbb{Q} \arrow[ur, dash, "\cong \mathbb{Z}/3^n\mathbb{Z}"'] & \end{tikzcd}$$

· $\mathbb{B}_{3,n} = \mathbb{Q}(2\cos\frac{2\pi}{3^{n+1}}) = \mathbb{B}_n$

◎ $\mathbb{B}_{p,\infty} = \bigcup_n \mathbb{B}_{p,n}$ : cyclo. $\mathbb{Z}_p$-ext$/\mathbb{Q}$

$\mathrm{Gal}(\mathbb{B}_{p,n}/\mathbb{Q}) \cong \mathbb{Z}_p$

$h_{p,n} = h(\mathbb{B}_{p,n})$

**Weber's Problem** $h_{p,n} = 1 \ (\forall n)$ ?

◎ · $p = 2$, $n = 1, \dots, 5$
· $p = 3$, $n = 1, 2, 3$
· $p = 5$, $n = 1$
$\Big\} \Rightarrow h_{p,n} = 1$

**W.P.** $\Leftrightarrow \forall \ell$ : prime number $\ell \nmid h_{p,n} \ (\forall n)$ ?

◎ · $\ell = p$ [Iwasawa] $p \nmid h_{p,n} \ (\forall n)$

· $p = 2$ [Horie] $\ell \not\equiv \pm 1 \ (8) \Rightarrow \ell \nmid h_{2,n}$

[Fukuda-Komatsu] $\ell \not\equiv \pm 1 \ (32) \Rightarrow$ "

$\ell < 5 \times 10^8 \Rightarrow$ "

- $p=3$ [Horie] $\ell \not\equiv \pm 1 \ (9) \Rightarrow \ell \nmid h_{3,n}$

  [M] $\ell < 4 \times 10^5 \Rightarrow$ 〃

## §3 Main Th.

Notation $p$ : prime, $\ell$ : prime $\neq p$

$$q = \begin{cases} 4 & (p=2) \\ p & (p \geq 3) \end{cases}$$

$f$ : inertia deg. of $\ell$ in $\mathbb{Q}(\mu_q)$

$s$ : $p^s \parallel \ell^f - 1$

$c := \varphi(p^s) = [\mathbb{Q}(\mu_{p^s}) : \mathbb{Q}]$

Th. (Horie)

$^\exists H(p,s,f) \in \mathbb{R}$

s.t. $\ell \nmid h_{p,s-1}$, $\ell > H(p,s,f) \Rightarrow \ell \nmid h_{p,n}$ $(\forall n)$ □

Th. 2

$$G(p,s,f) = \begin{cases} (c!)^{1/f} & (p=2) \\ (\sqrt{2}^{\,c} \cdot c!)^{1/f} & (p=3) \\ \left(\left(\frac{\sqrt{6p}}{2}\right)^c \cdot c!\right)^{1/f} & \end{cases}$$

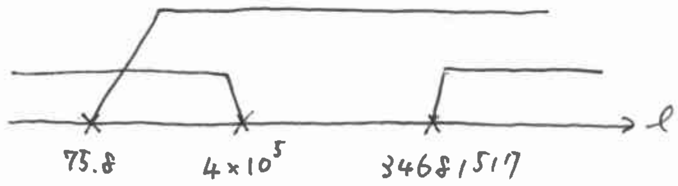
$\ell > G(p,s,f) \Rightarrow \ell \nmid h_{p,n}$ $(\forall n)$ □

◎ $p=3$

$\ell \equiv 8, 17 \ (27) \Longleftrightarrow s=2, f=2$

$H(3,2,2) = 34681517$

$G(3,2,2) = 75.8 \cdots$




Similarly,

$\ell \equiv 10, 19 \ (27) \Rightarrow G(3,2,1) = 5760$

$\longrightarrow$ Th. 1.

◎ $p=3$

## §4

$\mathbb{B}_n = \mathbb{Q}\left(2\cos\frac{2\pi}{3^{n+1}}\right)$, $\zeta_n = \exp\left(\frac{2\pi\sqrt{-1}}{3^n}\right)$, $h_n = h(\mathbb{B}_n)$

$\sigma$ : $\mathrm{Gal}(\mathbb{B}_n/\mathbb{Q}) = \langle\sigma\rangle$ ; $\zeta_{n+1}^{\sigma} = \zeta_{n+1}^{4}$

$\tau := \sigma^{3^{n-1}}$ : $\mathrm{Gal}(\mathbb{B}_n/\mathbb{B}_{n-1}) = \langle\tau\rangle$

$E_n := \mathcal{O}_{\mathbb{B}_n}^{\times}$

$e_n$ : gp. of cyclotomic units of $\mathbb{B}_n$

$$1 \to E_{n-1}/e_{n-1} \to E_n/e_n \to E_n^{1-\tau}/e_n^{1-\tau} \to 1 .$$

※ [Sinnott] $h_n = (E_n : e_n)$

$$\Longrightarrow (E_n^{1-\tau} : e_n^{1-\tau}) = \frac{h_n}{h_{n-1}}$$

$$\eta_n := \frac{\zeta_{n+1} - \zeta_{n+1}^{-1}}{\zeta_1\zeta_{n+1} - \zeta_1^{-1}\zeta_{n+1}^{-1}} \quad : n\text{-th Horie unit} .$$

◎ $e_n^{1-\tau} = \langle \eta_n^{1-\sigma} \rangle_{\mathbb{Z}[\sigma]}$

$$\mathbb{Z}[\zeta_n] \xrightarrow{\sim} \mathbb{Z}[\sigma]/(1+\tau+\tau^2)$$

$$\alpha = \sum a_i \zeta_n^i \longmapsto \alpha_\sigma := \sum a_i \sigma^i \curvearrowright (B_n^\times)^{1-\tau}$$

$$E_n^{1-\tau} \subseteq (B_n^\times)^{1-\tau}$$

**Lem.** (Horie)

$\ell \mid \frac{h_n}{h_{n-1}} \iff \exists \mathcal{L}$ : prime ideal of $\mathbb{Q}(\zeta_5)$

s.t. $\mathcal{L} \mid \ell$. $\forall \alpha \in \ell\mathcal{L}^{-1}$. $\eta_n^{\alpha_\sigma} = \varepsilon^\ell$ ($\exists \varepsilon \in E_n$)

## §5

$N := 3^n$ , $\varepsilon \in E_n \setminus E_{n-1}$

**Def.** (Mahler measure of $\varepsilon$)

$$M(\varepsilon) := \prod_{i=0}^{N-1} \max\{1, |\varepsilon^{\sigma^i}|\}$$ □

◎ Assume $\exists n$, $\ell \mid \frac{h_n}{h_{n-1}}$

$\rightsquigarrow \exists \mathcal{L}$. $\forall \alpha \in \ell\mathcal{L}^{-1}$. $\exists \varepsilon$. $\eta_n^{\alpha_\sigma} = \varepsilon^\ell$

$\overset{MM}{\rightsquigarrow} M(\varepsilon)^\ell = M(\eta_n^{\alpha_\sigma})$

**Lem. 1** $\ell > G(p.s.f)$

$\Longrightarrow \exists \alpha \in \ell\mathcal{L}^{-1}$ s.t. $M(\eta_n^{\alpha_\sigma}) < \exp\left(\frac{\ell N}{2} \cdot 0.77\right)$ □

**Th.** (Schinzel)

$\varepsilon \in E_n \setminus E_{n-1}$ , $\varepsilon^2 - 1 \in \mathfrak{a}$ : ideal

$A = (N_r(\mathfrak{a}))^{1/N}$

then $M(\varepsilon) \geq \left(\frac{A+\sqrt{A^2+4}}{2}\right)^{\frac{N}{2}}$ □

※ $\forall \varepsilon$, $\mathfrak{a} = \mathcal{O}_{B_n}$ , $A = 1$

$$M(\varepsilon) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{N}{2}}$$

· $\mathfrak{p}$ : prime ideal of $B_n$. $\mathfrak{p} \mid 3$

**Lem. 2** $\varepsilon \in E_n$ , $N_{r\,B_n/B_{n-1}}(\varepsilon) = 1$

$\Longrightarrow \varepsilon^2 - 1 \in \mathfrak{p}^{\frac{N-1}{2}}$ $\quad (N = 3^n)$ □

<u>Proof</u> $\exists x \in \mathbb{Z}[\zeta_{n+1}]$ s.t. $\varepsilon = x^{1-\tau}$

$\mathfrak{P} \mid \mathfrak{q}$ in $\mathbb{Q}(\zeta_{n+1})$ $\qquad x = 1 + \varepsilon + \varepsilon^{1+\tau}$

$$\mathrm{ord}_{\mathfrak{P}}(\varepsilon - 1) = \mathrm{ord}_{\mathfrak{P}}\left(\frac{x - x^{\tau}}{x^{\tau}}\right) \geq N - 2$$

$$\rightsquigarrow \mathrm{ord}_{\mathfrak{q}}(\varepsilon - 1) \geq \frac{N-2}{2} \quad //$$

<u>Lem. 3</u> $\varepsilon : Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$

$$M(\varepsilon) \geq \left(\frac{3^{\frac{N-1}{2N}} + \sqrt{3^{\frac{N-1}{N}} + 4}}{2}\right)^{\frac{N}{2}} \qquad \square$$

<u>Assume</u> $l \mid \frac{h_n}{h_{n-1}}$

$h_1 = h_2 = h_3 = 1 \Longrightarrow n \geq 4$

$$M(\varepsilon) \geq \left(\frac{3^{\frac{40}{81}} + \sqrt{3^{\frac{80}{81}} + 4}}{2}\right)$$

$$> \exp\left(0.778 \cdot \frac{N}{2}\right)$$

<u>Assume</u> $l > G(p.s.f)$

$$\exp\left(0.778 \cdot \frac{Nl}{2}\right) < M(\varepsilon)^{l} = M(\eta_n^{\alpha_0}) < \exp\left(0.77 \cdot \frac{N}{2}\right)$$

$$\rightsquigarrow 0.778 < 0.77$$

$l > G(p.s.f)$

$$\Rightarrow l \nmid \frac{h_n}{h_{n-1}} \ (\forall n) \quad \Rightarrow l \nmid h_n \ (\forall n)$$