Some Topics Related to Modular Differential Equations

(保型微分方程式入門)

Yuichi SAKAI

Some Topics Related to Modular Differential Equations

1. What is a modular differential equation ?

2. On modular solutions of certain meromorphic modular differential equations.

3. The Atkin orthogonal polynomials for the low-level Fricke groups and their applications.

1. Modular differential equation

$$f''(\tau) + C_1(\tau)f'(\tau) + C_2(\tau)f(\tau) = 0$$

$$\tau \in \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}, \quad k \in \mathbb{Q}_{\geq 0}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{PSL}_2(\mathbb{R}): \text{ covolume fini}$$

$$f(\tau) \text{ is its solution} \Longrightarrow \quad (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \text{ is also its solution.}$$

Its solution space is invariant w.r.t. the action for Γ

1. Modular differential equation

$$f''\Big(\frac{a\tau+b}{c\tau+d}\Big)+C_1\Big(\frac{a\tau+b}{c\tau+d}\Big)f'\Big(\frac{a\tau+b}{c\tau+d}\Big)+C_2\Big(\frac{a\tau+b}{c\tau+d}\Big)f\Big(\frac{a\tau+b}{c\tau+d}\Big)=0$$

$$C_1\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^2C_1(\tau)-\frac{k+1}{\pi\sqrt{-1}}c(c\tau+d)$$

$$C_2\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^4C_2(\tau)-\frac{k}{2\pi\sqrt{-1}}c(c\tau+d)^3C_1(\tau)+\frac{k(k+1)}{(2\pi\sqrt{-1})^2}c^2(c\tau+d)^2$$

$$E_2\Big(\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^2E_2(\tau)+\frac{6}{\pi\sqrt{-1}}c(c\tau+d)\quad\text{for}\quad\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$$

$$E_2(\tau)=1-24\sum_{n=1}^{\infty}\Big(\sum_{d|n}d\Big)q^n\qquad(q=e^{2\pi\sqrt{-1}\tau}).$$

1. Modular differential equation

$$f''\left(\frac{a\tau+b}{c\tau+d}\right)+C_1\left(\frac{a\tau+b}{c\tau+d}\right)f'\left(\frac{a\tau+b}{c\tau+d}\right)+C_2\left(\frac{a\tau+b}{c\tau+d}\right)f\left(\frac{a\tau+b}{c\tau+d}\right)=0$$

$$C_1\left(\frac{a\tau+b}{c\tau+d}\right)+\frac{k+1}{6}E_2\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2\left(C_1(\tau)+\frac{k+1}{6}E_2(\tau)\right)$$

$$C_2\left(\frac{a\tau+b}{c\tau+d}\right)-\frac{k(k+1)}{12}E_2'\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^4\left(C_2(\tau)-\frac{k(k+1)}{12}E_2'(\tau)\right)$$

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2E_2(\tau)+\frac{6}{\pi\sqrt{-1}}c(c\tau+d)\quad \text{for}\quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$$

$$E_2(\tau)=1-24\sum_{n=1}^{\infty}\Big(\sum_{d|n}d\Big)q^n \qquad (q=e^{2\pi\sqrt{-1}\tau}).$$

1. Modular differential equation

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

Kaneko-Zagier Equation

$$k = 4 \qquad \Longrightarrow \qquad E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\Big(\sum_{d|n} d^3\Big)q^n \quad (q = e^{2\pi\sqrt{-1}\tau})$$

$$k = 6 \qquad \Longrightarrow \qquad E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\Big(\sum_{d|n} d^5\Big)q^n$$

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728}$$

1. Modular differential equation

$$f''(\tau) - \frac{k+1}{6} E_2(\tau) f'(\tau) + \frac{k(k+1)}{12} E_2'(\tau) f(\tau) = 0$$

Kaneko-Zagier Equation

$$f(\tau) = E_4(\tau)^\alpha E_6(\tau)^\beta \Delta(\tau)^m \widetilde{f}(j) \quad \Longrightarrow \quad ss_p(j) \equiv j^\alpha (j-1728)^\beta \widetilde{f}(j) \pmod p$$

$$p \geq 5: \text{ prime}, \; k = p-1, \quad m = \left[\frac{p}{12}\right], \; k = 12m + 4\alpha + 6\beta$$

$$ss_p(j) = \prod_{\substack{E/\overline{\mathbb{F}}_p \\ E: supersingular}} (j - j(E)) \in \mathbb{F}_p[j]$$

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}$$

1. Modular differential equation

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

$$\widetilde{f}(j) = \binom{m - \frac{k+1}{6}}{m} j^m F\Big(-m, \frac{1-2\alpha}{3}, 1 - \frac{k+1}{6}; \frac{1728}{j}\Big)$$

$$p \geq 5: \ \text{prime}, \ k = p-1, \quad m = \left[\frac{p}{12}\right], \ k = 12m + 4\alpha + 6\beta$$

$$F(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \binom{-\alpha}{n}\binom{-\beta}{n}\binom{-\gamma}{n}^{-1}(-x)^n \quad : \quad \text{The Gauss hypergeometric series}$$

$$x(1-x)\frac{d^2F}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dF}{dx} - \alpha\beta F = 0$$

1. Modular differential equation

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

$$ss_p(j) = j^{\alpha}(j-1728)^{\beta}\begin{cases} j^m F\left(\frac{1}{12}, \frac{5}{12}, 1, \frac{1728}{j}\right) & p \equiv 1, 5 \pmod{12} \\ j^m F\left(\frac{7}{12}, \frac{11}{12}, 1, \frac{1728}{j}\right) & p \equiv 7, 11 \pmod{12} \end{cases} \pmod{p}$$

$$p \geq 5: \text{ prime}, \ k = p-1, \quad m = \left[\frac{p}{12}\right], \ k = 12m + 4\alpha + 6\beta$$

$$F(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \binom{-\alpha}{n}\binom{-\beta}{n}\binom{-\gamma}{n}^{-1}(-x)^n \ : \quad \text{The Gauss hypergeometric series}$$

$$x(1-x)\frac{d^2F}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dF}{dx} - \alpha\beta F = 0$$

1. Modular differential equation

$$f''(\tau) - \frac{k+1}{6} E_2(\tau) f'(\tau) + \frac{k(k+1)}{12} E_2'(\tau) f(\tau) = 0$$

$$f(\tau) = \begin{cases} E_4(\tau)^{\frac{k}{4}} F\Big(-\frac{k}{12}, -\frac{k-4}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\Big) & \text{for} \quad k \equiv 0,\ 4 \pmod{12}, \\ E_4(\tau)^{\frac{k-6}{4}} E_6(\tau) F\Big(-\frac{k-6}{12}, -\frac{k-10}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\Big) & \text{for} \quad k \equiv 6,\ 10 \pmod{12}. \end{cases}$$

$$F(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \binom{-\alpha}{n} \binom{-\beta}{n} \binom{-\gamma}{n}^{-1} (-x)^n \ : \quad \text{The Gauss hypergeometric series}$$

$$x(1-x)\frac{d^2F}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dF}{dx} - \alpha\beta F = 0$$

1. Modular differential equation

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

The type of solution	Groups	Weights
Hypergeometric	$\mathrm{SL}_2(\mathbb{Z})$	$k \equiv 0,\ 4 \pmod 6$
Hypergeometric	$\Gamma_0(2),\quad \Gamma(2)$	$k \equiv 2 \pmod 6$
Hypergeometric	$\Gamma_0(3),\quad \Gamma_0^0(3)$	$k \equiv 1,\ 3 \pmod 6$
Hypergeometric	$\Gamma_0(4),\quad \Gamma_0^0(4)$	$k \equiv \frac{1}{2} \pmod 3$
Heun	$\Gamma(5)$	$k = \frac{6n+1}{5} \notin \mathbb{Z}$

1. Modular differential equation

H. Tsutsumi

$$f''(\tau) + C_1(\tau)f'(\tau) + C_2(\tau)f(\tau) = 0$$

$$C_1(\tau) = \frac{s_1E_4(\tau)^3 + s_2E_6(\tau)^2}{E_4(\tau)E_6(\tau)} - \frac{k+1}{6}E_2(\tau)$$

$$C_2(\tau) = \frac{k(k+1)}{12}E_2'(\tau) - \frac{k}{12}\frac{(s_1E_4(\tau)^3 + s_2E_6(\tau)^2)'}{E_4(\tau)E_6(\tau)} + \frac{\Delta(\tau)(s_3E_4(\tau)^3 - s_4E_6(\tau)^2)}{E_4(\tau)^2E_6(\tau)^2} \quad \text{for } s_1, s_2, s_3, s_4 \in \mathbb{C}.$$

This MDE has modular solutions of hypergeometric type for $\mathrm{SL}_2(\mathbb{Z})$

(with some conditions)

1. Modular differential equation

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

The type of solution	Groups	Weights
Hypergeometric	$\mathrm{SL}_2(\mathbb{Z})$	$k \equiv 0,\ 4 \pmod 6$
Hypergeometric	$\Gamma_0(2), \quad \Gamma(2)$	$k \equiv 2 \pmod 6$
Hypergeometric	$\Gamma_0(3), \quad \Gamma_0^0(3)$	$k \equiv 1,\ 3 \pmod 6$
Hypergeometric	$\Gamma_0(4), \quad \Gamma_0^0(4)$	$k \equiv \frac{1}{2} \pmod 3$
Heun	$\Gamma(5)$	$k = \frac{6n+1}{5} \notin \mathbb{Z}$

1. MDE

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

$k \equiv 5 \pmod 6$

$$\sqrt{\Delta(\tau)}^n P_n\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right)\frac{E_4'(\tau)}{240} - \sqrt{\Delta(\tau)}^{n+1} Q_n\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right) = O(q^{n+1})$$

is a solution of its MDE of weight $k+1 = 6(n+1)$ $(n = 0, 1, \ldots)$

Extremal quasimodular forms

$$P_0(X) = 1, \quad P_1(X) = X, \quad P_{n+1}(X) = XP_n(X) - 12\frac{(6n+1)(6n+5)}{n(n+1)}P_{n-1}(X)$$

$$Q_0(X) = 0, \quad Q_1(X) = 1, \quad Q_{n+1}(X) = XQ_n(X) - 12\frac{(6n+1)(6n+5)}{n(n+1)}Q_{n-1}(X)$$

1. MDE

Extremal quasimodular forms

$$f(\tau) = f_0(\tau) + f_1(\tau)E_2(\tau) + \cdots + f_r(\tau)E_2(\tau)^r \in QM_k^{(r)}(\mathrm{SL}_2(\mathbb{Z}))$$

$$f_i(\tau) \in M_{k-2i}(\mathrm{SL}_2(\mathbb{Z}))$$

$$m = \dim_{\mathbb{C}} QM_k^{(r)}(\mathrm{SL}_2(\mathbb{Z})) = \sum_{i=0}^{r} \dim_{\mathbb{C}} M_{k-2i}(\mathrm{SL}_2(\mathbb{Z}))$$

$$ord_{\sqrt{-1}\infty}(f) = m - 1 \qquad \text{i.e.} \quad f(\tau) = O(q^{m-1})$$

Extremal quasimodular forms(ex-qmf.)

1. MDE Extremal quasimodular forms

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

$k \equiv 5 \pmod 6$

$$\sqrt{\Delta(\tau)}^{n-1}P_{n-1}\Big(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\Big)\frac{E_4'(\tau)}{240} - \sqrt{\Delta(\tau)}^{n}Q_{n-1}\Big(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\Big) = O(q^n)$$

is a solution of its MDE of weight $k+1 = 6n \ (n = 1, 2, \ldots)$

$$n = ord_{\sqrt{-1}\infty} f = \dim_{\mathbb{C}} M_{k+1} + \dim_{\mathbb{C}} M_{k-1} - 1$$

$$P_0(X) = 1, \quad P_1(X) = X, \quad P_{n+1}(X) = XP_n(X) - 12\frac{(6n+1)(6n+5)}{n(n+1)}P_{n-1}(X)$$

$$Q_0(X) = 0, \quad Q_1(X) = 1, \quad Q_{n+1}(X) = XQ_n(X) - 12\frac{(6n+1)(6n+5)}{n(n+1)}Q_{n-1}(X)$$

1. MDE Extremal quasimodular forms

$$f''(\tau) - \Big(\frac{k+1}{6}E_2(\tau) - \frac{1}{3}\frac{E_6(\tau)}{E_4(\tau)}\Big)f'(\tau) + \Big(\frac{k(k+1)}{12}E_2'(\tau) - \frac{k}{18}\frac{E_6'(\tau)}{E_4(\tau)}\Big)f(\tau) = 0$$

$k \equiv 1 \pmod 6$

$$\sqrt{\Delta(\tau)}^{n-1} P_{n-1}^*\Big(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\Big)\Big(-\frac{E_6'(\tau)}{504}\Big) - \sqrt{\Delta(\tau)}^{n} Q_{n-1}^*\Big(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\Big)E_2(\tau) = O(q^n)$$

is a solution of its MDE of weight $k+1 = 6n+2$ $(n = 1, 2, \ldots)$

$$n = ord_{\sqrt{-1}\infty} f = \dim_{\mathbb{C}} M_{k+1} + \dim_{\mathbb{C}} M_{k-1} - 1$$

$$P_0^*(X) = 1, \quad P_1^*(X) = X, \quad P_{n+1}^*(X) = XP_n^*(X) - 12\frac{(6n-1)(6n+7)}{n(n+1)}P_{n-1}^*(X)$$

$$Q_0^*(X) = 0, \quad Q_1^*(X) = 1, \quad Q_{n+1}^*(X) = XQ_n^*(X) - 12\frac{(6n-1)(6n+7)}{n(n+1)}Q_{n-1}^*(X)$$

1. MDE

Extremal quasimodular forms

$$f''(\tau) - \Big(\frac{k+1}{6}E_2(\tau) - \frac{2}{3}\frac{E_6(\tau)}{E_4(\tau)}\Big)f'(\tau) + \Big(\frac{k(k+1)}{12}E_2'(\tau) - \frac{k}{9}\frac{E_6'(\tau)}{E_4(\tau)} - \frac{2}{9}\Big(E_4(\tau) - \frac{E_6(\tau)^2}{E_4(\tau)^2}\Big)\Big)f(\tau) = 0$$

$k \equiv 3 \pmod 6$

$$E_4(\tau)\left(\sqrt{\Delta(\tau)}^{\,n-1}P_{n-1}\Big(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\Big)\frac{E_4'(\tau)}{240} - \sqrt{\Delta(\tau)}^{\,n}Q_{n-1}\Big(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\Big)\right) = O(q^n)$$

is a solution of its MDE of weight $k+1 = 6n+4$ $(n = 1, 2, \dots)$

$$n = ord_{\sqrt{-1}\infty} f = \dim_{\mathbb{C}} M_{k+1} + \dim_{\mathbb{C}} M_{k-1} - 1$$

$$P_0(X) = 1, \quad P_1(X) = X, \quad P_{n+1}(X) = XP_n(X) - 12\frac{(6n+1)(6n+5)}{n(n+1)}P_{n-1}(X)$$

$$Q_0(X) = 0, \quad Q_1(X) = 1, \quad Q_{n+1}(X) = XQ_n(X) - 12\frac{(6n+1)(6n+5)}{n(n+1)}Q_{n-1}(X)$$

2. On modular solutions

$f(\tau)$

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

$\frac{f(\tau)}{E_4(\tau)}$

$$f'' - \left(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\right)f' + \left(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\right)f = 0$$

$E_4(\tau)f(\tau)$

$$f'' - \left(\frac{k+1}{6}E_2 - \frac{2}{3}\frac{E_6}{E_4}\right)f' + \left(\frac{k(k+1)}{12}E_2' - \frac{k}{9}\frac{E_6'}{E_4} - \frac{2}{9}\left(E_4 - \frac{E_6^2}{E_4^2}\right)\right)f = 0$$

$f(\tau)$

Such MDE has modular solution of hypergeometric type for $\mathrm{SL}_2(\mathbb{Z})$ (proved by H. Tsutsumi)

2. On modular solutions

$$f'' - \left(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\right)f' + \left(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\right)f = 0$$

For $\mathrm{SL}_2(\mathbb{Z})$

$$E_4(\tau)^{\frac{k}{4}}F\left(-\frac{k}{12}, -\frac{k-8}{12}, -\frac{k-7}{6} ; \frac{1728}{j(\tau)}\right) = 1 + O(q) \quad \text{if } k \equiv 0,\ 8 \pmod{12},$$

$$E_4(\tau)^{\frac{k-6}{4}}E_6(\tau)F\left(-\frac{k-6}{12}, -\frac{k-14}{12}, -\frac{k-7}{6} ; \frac{1728}{j(\tau)}\right) = 1 + O(q) \quad \text{if } k \equiv 2,\ 6 \pmod{12}.$$

Other modular solutions of its MDE ?

2. On modular solutions

$$Hl(a,w;\alpha,\beta,\gamma,\delta;x)=\sum_{n=0}^{\infty}c_n x^n$$

: Heun local series

$$c_0=1,\quad c_1=\tfrac{w}{a\gamma}c_0,\ \gamma+\delta+\varepsilon=\alpha+\beta+1,$$

$$c_{n+1}=\frac{(n[(n-1+\gamma)(1+a)+a\delta+\varepsilon]+w)}{(n+1)(n+\gamma)a}c_n-\frac{(\gamma-1+\alpha)(\gamma-1+\beta)}{(n+1)(n+\gamma)a}c_{n-1}\qquad(n\geq 1)$$

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\varepsilon}{x-a}\right)\frac{dy}{dx}+\frac{\alpha\beta x-w}{x(x-1)(x-a)}y=0.$$

$x^{1-\gamma}Hl(a,w';\alpha',\beta',\gamma',\delta;x)$ is another solution

$\alpha'=\alpha+1-\gamma,\ \beta'=\beta+1-\gamma,\ \gamma'=2-\gamma,\ w'=(a\delta+\varepsilon)(1-\gamma)+w$

2. On modular solutions

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

For level 5, $\quad k = \frac{6n+1}{5} \notin \mathbb{Z}$

$$\phi_1^{5k} Hl\left(\frac{\xi_2}{\xi_1}, \frac{k(1-5k)}{2\xi_1}; -k, \frac{1-5k}{6}, -\frac{k-5}{6}, \frac{1-5k}{6}; \frac{\phi_2^5\phi_1^{-5}}{\xi_1}\right) = 1 + O(q),$$

$$\phi_2^{\frac{5(k+1)}{6}}\phi_1^{\frac{5(5k-1)}{6}} Hl\left(\frac{\xi_2}{\xi_1}, \frac{(5k-1)(11-7k)}{36\xi_1}; \frac{1-5k}{6}, \frac{1-2k}{3}, \frac{k+7}{6}, \frac{1-5k}{6}; \frac{\phi_2^5\phi_1^{-5}}{\xi_1}\right) = q^{\frac{k+1}{6}} + O(q^{\frac{k+7}{6}}),$$

$$\phi_1(\tau) = \frac{1}{\eta(\tau)^{3/5}}\sum_{n\in\mathbb{Z}}(-1)^n q^{(10n+1)^2/40}, \quad \phi_2(\tau) = \frac{1}{\eta(\tau)^{3/5}}\sum_{n\in\mathbb{Z}}(-1)^n q^{(10n+3)^2/40}$$

$$\xi_1 = -\tfrac{11+5\sqrt{5}}{2}, \;\; \xi_2 = \xi_1 + 5\sqrt{5}$$

2. On modular solutions

$$f'' - \Big(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\Big)f' + \Big(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\Big)f = 0$$

For level 2, $k \equiv 4 \pmod 6$

$$H_2(\tau)^{\frac{k}{2}} Hl\left(-\frac{1}{3}, -\frac{k(k-10)}{36}; -\frac{k}{4}, -\frac{k-2}{4}, -\frac{k-7}{6}, -\frac{k-4}{3}; \frac{64}{j_2(\tau)}\right) = 1 + O(q),$$

$$\Delta_2(\tau)^{\frac{k-1}{6}} H_2(\tau)^{\frac{k+2}{6}} Hl\left(-\frac{1}{3}, -\frac{5k^2-22k-10}{108}; -\frac{k+2}{12}, -\frac{k-4}{12}, \frac{k+5}{6}, -\frac{k-4}{3}; \frac{64}{j_2(\tau)}\right) = q^{\frac{k-1}{6}} + O(q^{\frac{k+5}{6}}),$$

$$\Gamma_0(2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z});\quad c \equiv 0 \pmod 2\right\} \qquad \Gamma(2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2);\quad b \equiv 0,\ a \equiv d \equiv 1 \pmod 2\right\}$$

$$H_2(\tau) = 1 + 24\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d|n \\ d:\mathrm{odd}}}\Bigg)q^n, \quad \Delta_2(\tau) = \frac{\eta(2\tau)^{16}}{\eta(\tau)^8}, \quad j_2(\tau) = \frac{H_2(\tau)^2}{\Delta_2(\tau)}$$

2. On modular solutions

$$f'' - \Big(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\Big)f' + \Big(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\Big)f = 0$$

For level 3, $\quad k \equiv 3,\ 5 \pmod 6$

$$I_3(\tau)^k Hl\left(-\frac{1}{8}, -\frac{k(k-11)}{72}; -\frac{k}{3}, -\frac{k-2}{3}, -\frac{k-7}{6}, \frac{k-3}{2}; \frac{27}{j_3(\tau)}\right) = 1 + O(q)$$

$$\Delta_3(\tau)^{\frac{k-1}{6}} I_3(\tau)^{\frac{k+1}{2}} Hl\left(-\frac{1}{8}, \frac{k^2-2k-39}{288}; -\frac{k+1}{6}, -\frac{k-3}{6}, \frac{k+5}{6}, \frac{k-3}{2}; \frac{27}{j_3(\tau)}\right) = q^{\frac{k-1}{6}} + O(q^{\frac{k+5}{6}})$$

$$\Gamma_0(3) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z});\quad c \equiv 0 \pmod 3\right\} \qquad \Gamma_0^0(3) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(3);\quad b \equiv 0 \pmod 3\right\}$$

$$I_3(\tau) = 1 + 6\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{d}{3}\right)\right)q^n, \quad \Delta_3(\tau) = \frac{\eta(3\tau)^9}{\eta(\tau)^3}, \quad j_3(\tau) = \frac{I_3(\tau)^3}{\Delta_3(\tau)}$$