2. On modular solutions

$$f'' - \Big(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\Big)f' + \Big(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\Big)f = 0$$

For level 4, $\quad k = 3n - \frac{1}{2} \equiv \frac{5}{2} \pmod 3$

$$P_{n+2}^{(4)}(X) := (1 - 33X - 33X^2 + X^3)P_n^{(4)}(X) + \lambda_n^{(4)} X(1 - 4X + 6X^2 + X^4)P_{n-2}^{(4)}(X) \quad (n \geq 3)$$

$$Q_{n+2}^{(4)}(X) := \frac{-1}{\lambda_{n+2}^{(4)}}\Big(16(1 - 33X - 33X^2 + X^3)Q_n^{(4)}(X) - X(1 - 4X + 6X^2 + X^4)Q_{n-2}^{(4)}(X)\Big) \quad (n \geq 3)$$

$P_1^{(4)}(X) = 1 - 5X,\qquad P_2^{(4)}(X) = 1 - 22X - 11X^2,$

$P_3^{(4)}(X) = 1 + 17X + 187X^2 + 51X^3,\qquad P_4^{(4)}(X) = 1 + 506X^2 + 1228X^3 + 253X^4,$

$Q_1^{(4)}(X) = 1 - \frac{X}{5},\qquad Q_2^{(4)}(X) = 1 + 2X - \frac{X^2}{11},$

$Q_3^{(4)}(X) = 1 + \frac{11}{3}X + \frac{X^2}{3} + \frac{X^3}{51},\qquad Q_4^{(4)}(X) = 1 + \frac{56}{11}X + 2X^2 + \frac{X^4}{253},$

$$\lambda_n^{(4)} = 48\frac{(6n-1)(6n-17)}{4n^2 - 12n + 5}$$

2. On modular solutions

$$f'' - \Big(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\Big)f' + \Big(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\Big)f = 0$$

For level 4, $k = 3n - \frac{1}{2} \equiv \frac{5}{2} \pmod 3$

$$\theta(\tau)^{2k} P_n^{(4)}\Big(\frac{16}{j_4(\tau)}\Big) = 1 + O(q)$$

$$\Delta_4(\tau)^{\frac{k-1}{6}}\theta(\tau)^{\frac{2(2k+1)}{3}} Q_n^{(4)}\Big(\frac{16}{j_4(\tau)}\Big) = q^{\frac{k-1}{6}} + O(q^{\frac{k+5}{6}})$$

$$\Gamma_0(4) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z});\quad c \equiv 0 \pmod 4\right\} \qquad \Gamma_0^0(4) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(4);\quad b \equiv 0 \pmod 4\right\}$$

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}, \quad \Delta_4(\tau) = \frac{\eta(4\tau)^8}{\eta(2\tau)^4}, \quad j_4(\tau) = \frac{\theta(\tau)^4}{\Delta_4(\tau)}$$

2. On modular solutions

$$f'' - \left(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\right)f' + \left(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\right)f = 0$$

For level 5, $k = \frac{6n+5}{5} \notin \mathbb{Z}$

$$\begin{aligned} P_{n+5}^{(5)}(X) &:= (1+X^2)(1-522X-10006X^2+522X^3+X^4)P_n^{(5)}(X) \\ &\quad + \lambda_n^{(5)} X(1-11X-X^2)^5 P_{n-5}^{(5)}(X) \qquad (n \geq 6, n \not\equiv 0 \mod 5), \\ Q_{n+5}^{(5)}(X) &:= \frac{-1}{\lambda_{n+5}^{(5)}}\Big((1+X^2)(1-522X-10006X^2+522X^3+X^4)Q_n^{(5)}(X) \\ &\quad + X(1-11X-X^2)^5 Q_{n-5}^{(5)}(X)\Big) \qquad (n \geq 6, n \not\equiv 0 \mod 5) \end{aligned}$$

$$\begin{aligned} P_1^{(5)}(X) &= 1-66X-11X^2, \\ P_2^{(5)}(X) &= 1-119X+187X^2-17X^3, \\ P_3^{(5)}(X) &= 1-207X-391X^2-1173X^3+46X^4, \\ P_4^{(5)}(X) &= 1-435X-6670X^2-3335X^4+87X^5, \\ P_6^{(5)}(X) &= 1+369X+50594X^2+261580X^3 \\ &\quad +136735X^4-151003X^5+54858X^6-902X^7, \\ P_7^{(5)}(X) &= 1+141X+55037X^2+740673X^3+667870X^4 \\ &\quad +932292X^5-1640958X^6+340374X^7-4794X^8, \\ P_8^{(5)}(X) &= 1+53X+80454X^2+2277092X^3+6441196X^4 \\ &\quad +4589482X^5-6686904X^6+9218078X^7-1278731X^8+15847X^9, \\ P_9^{(5)}(X) &= 1+162545X^2+8777430X^3+57609370X^4+48470919X^5 \\ &\quad +29482300X^6-34622085X^7+28804685X^8-2925810X^9+32509X^{10}, \end{aligned}$$

$$\lambda_n^{(5)} = 12\frac{(6n+5)(6n-35)}{n(n-5)}$$

$$\begin{aligned} Q_1^{(5)}(X) &= -\frac{1}{11}P_1^{(5)}(-X), & Q_2^{(5)}(X) &= \frac{1}{17}P_2^{(5)}(-X), \\ Q_3^{(5)}(X) &= \frac{1}{46}P_3^{(5)}(-X), & Q_4^{(5)}(X) &= -\frac{1}{87}P_4^{(5)}(-X), \\ Q_6^{(5)}(X) &= \frac{1}{902}P_6^{(5)}(-X), & Q_7^{(5)}(X) &= -\frac{1}{4794}P_7^{(5)}(-X), \\ Q_8^{(5)}(X) &= -\frac{1}{15847}P_8^{(5)}(-X), & Q_9^{(5)}(X) &= \frac{1}{32509}P_9^{(5)}(-X), \end{aligned}$$

2. On modular solutions

$$f'' - \Big(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\Big)f' + \Big(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\Big)f = 0$$

For level 5, $\quad k = \dfrac{6n+5}{5} \notin \mathbb{Z}$

$$\phi_1(\tau)^{5k} P_n^{(5)}\Big(\frac{\phi_2(\tau)^5}{\phi_1(\tau)^5}\Big) = 1 + O(q),$$
$$\phi_2(\tau)^{5k} Q_n^{(5)}\Big(\frac{\phi_1(\tau)^5}{\phi_2(\tau)^5}\Big) = q^{\frac{k-1}{6}} + O(q^{\frac{k+5}{6}}),$$

$$\Gamma(5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z});\quad a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod 5 \right\}$$

$$\phi_1(\tau) = \frac{1}{\eta(\tau)^{3/5}} \sum_{n\in\mathbb{Z}} (-1)^n q^{(10n+1)^2/40}, \quad \phi_2(\tau) = \frac{1}{\eta(\tau)^{3/5}} \sum_{n\in\mathbb{Z}} (-1)^n q^{(10n+3)^2/40}$$

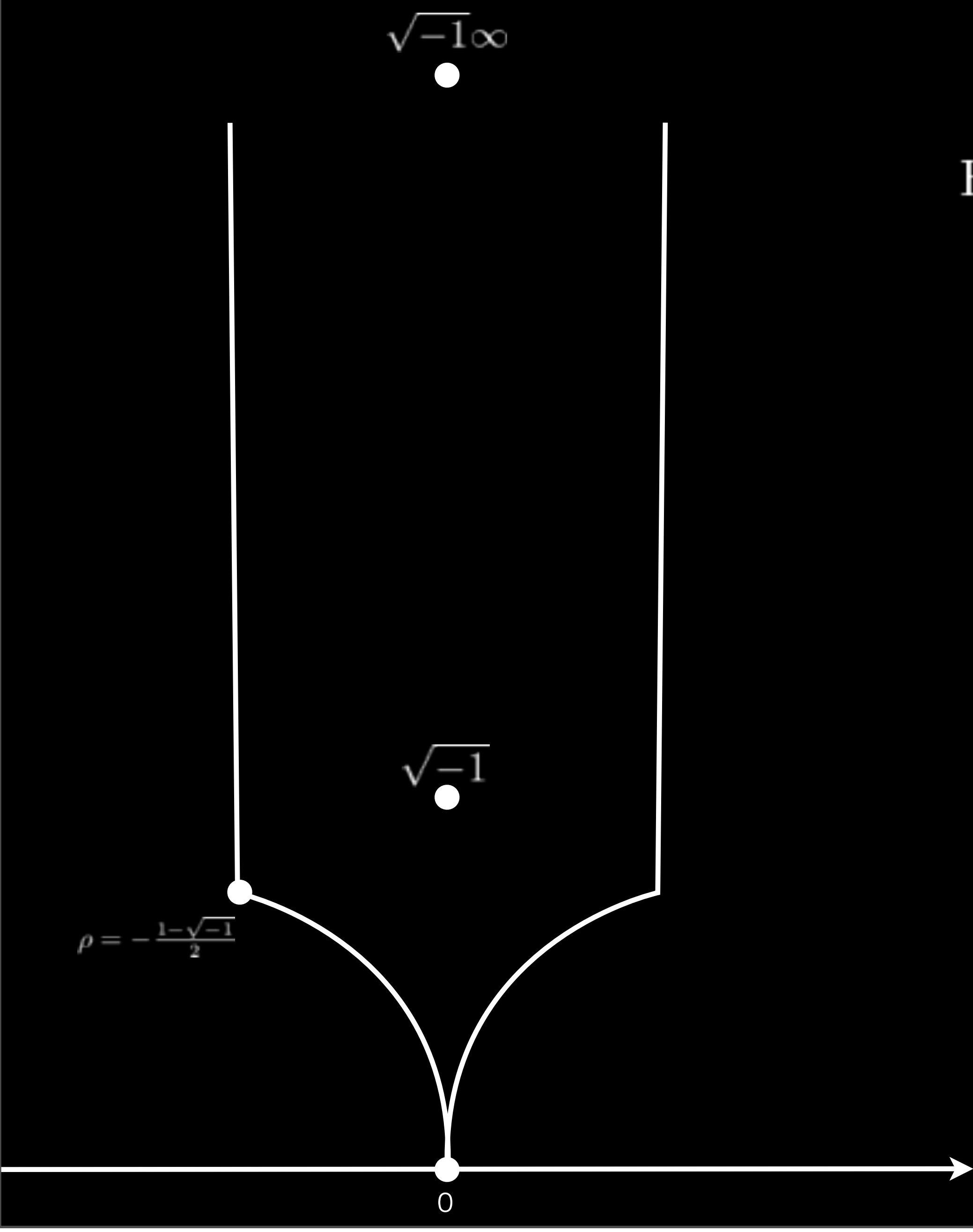
2. On modular solutions

$$f'' - \left(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\right)f' + \left(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\right)f = 0$$

The type of solution	Groups	Weights
Hypergeometric	$SL_2(\mathbb{Z})$	$k \equiv 0,\ 2 \pmod 6$
Heun	$\Gamma_0(2),\quad \Gamma(2)$	$k \equiv 4 \pmod 6$
Heun	$\Gamma_0(3),\quad \Gamma_0^0(3)$	$k \equiv 3,\ 5 \pmod 6$
Only recursion	$\Gamma_0(4),\quad \Gamma_0^0(4)$	$k \equiv \frac{5}{2} \pmod 3$
Only recursion	$\Gamma(5)$	$k = \frac{6n+5}{5} \notin \mathbb{Z}$

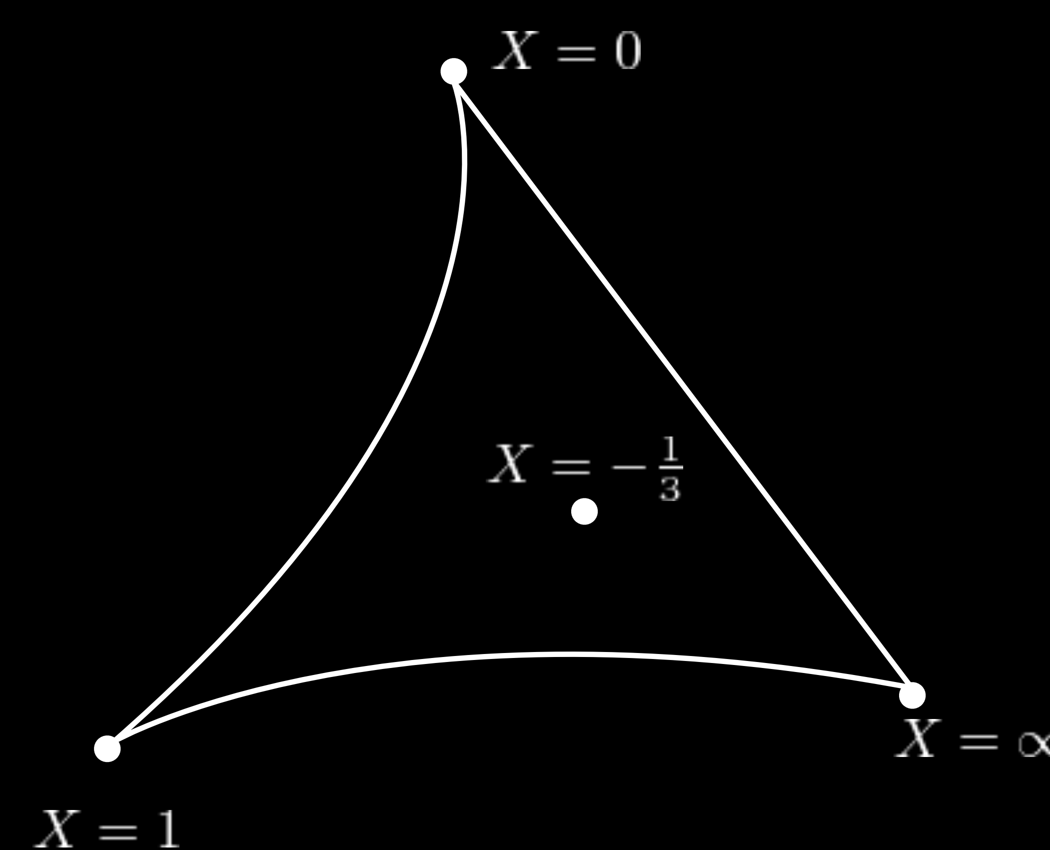
2. On modular solutions

$$f'' - \Big(\frac{k+1}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\Big)f' + \Big(\frac{k(k+1)}{12}E_2' - \frac{k}{18}\frac{E_6'}{E_4}\Big)f = 0$$



For the case of $\Gamma_0(2)$

$$X = \frac{64}{j_2(\tau)}$$



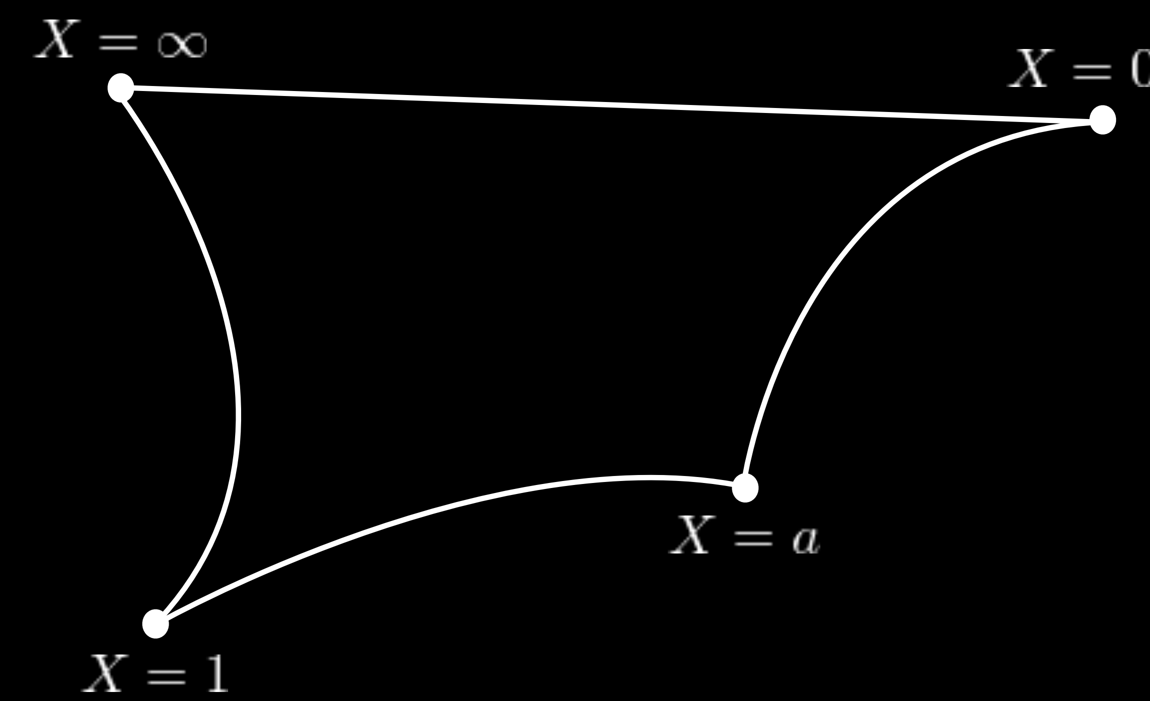
Arithmetic triangle group + One point = tetragonal

2. On modular solutions

Observation

2. On modular solutions

Observation



The MDE which has 4 regular singularities w.r.t. a local parameter $X=\dfrac{\alpha}{j_N(\tau)}$ seems to be Heun type.

Do all of MDEs with regular singularities at ell. pt. for $\mathrm{SL}_2(\mathbb{Z})$ have modular solution for level 1〜5 ?

What kind of MDE with regular singularities at ell. pt. have quasimodular solution ?

3. The Atkin orthogonal polynomials

Kaneko-Zagier

For $f,\ g \in \mathbb{C}[j]$ $\qquad (f,g)_{\mathrm{SL}_2(\mathbb{Z})} :=$ constant term of fgE_2 as a Laurent series in q

$$A_{n_p}^{(1)}(j) \equiv ss_p(j) \pmod p$$

$p \geq 5$: prime, $m = \left[\frac{p}{12}\right]$, $p-1 = 12m + 4\delta + 6\varepsilon$, $n_p = m + \delta + \varepsilon$

3. The Atkin orthogonal polynomials

Kaneko-Zagier

For $f,\ g \in \mathbb{C}[j]$ $\quad (f,g)_{\mathrm{SL}_2(\mathbb{Z})} :=$ constant term of fgE_2 as a Laurent series in q

: Non-degenerate

$$A^{(1)}_{n+1}(j) = \left(j - 24\frac{(144n^2-29)}{(2n-1)(2n+1)}\right)A^{(1)}_n(j) - 36\frac{(12n-13)(12n-7)(12n-5)(12n+1)}{n(n-1)(2n-1)^2}A^{(1)}_{n-1}(j)$$

$A^{(1)}_0(j) = 1,\ A^{(1)}_1(j) = j - 720$ and $A^{(1)}_2(j) = j^2 - 1640j + 269280$

$$A^{(1)}_{n_p}(j) \equiv ss_p(j) \pmod p$$

$p \geq 5$: prime, $m = \left[\frac{p}{12}\right]$, $p - 1 = 12m + 4\delta + 6\varepsilon$, $\quad n_p = m + \delta + \varepsilon$

3. The Atkin orthogonal polynomials

Kaneko-Zagier

For $f,\ g \in \mathbb{C}[j]$ $\quad (f,g)_{\mathrm{SL}_2(\mathbb{Z})} :=$ constant term of fgE_2 as a Laurent series in q

↓ : Non-degenerate

$$A_n^{(1)}(j) = \sum_{i=0}^{n} (-12)^{3i} \left[\sum_{m=0}^{i} \binom{-\frac{1}{12}}{i-m} \binom{-\frac{5}{12}}{i-m} \binom{n+\frac{1}{12}}{m} \binom{n-\frac{7}{12}}{m} \binom{2n-1}{m}^{-1} \right] j^{n-i}.$$

$$A_{n_p}^{(1)}(j) \equiv ss_p(j) \pmod{p}$$

$p \geq 5$: prime, $m = \left[\frac{p}{12}\right]$, $p-1 = 12m + 4\delta + 6\varepsilon$, $\quad n_p = m + \delta + \varepsilon$

3. The Atkin orthogonal polynomials

Kaneko-Koike

$$\Phi(j) := \frac{E_2(j)E_4(j)}{E_6(j)j} = \frac{B_n^{(1)}(j)}{A_n^{(1)}(j)} + O(j^{-2n-1})$$

$$\lambda_i = \begin{cases} 1 & n = 0 \\ 720 & n = 1 \\ 12\left(6 + \frac{(-1)^n}{n-1}\right)\left(6 + \frac{(-1)^n}{n}\right) & n \geq 2 \end{cases}$$

$$\prod_{i=0}^{2n} \lambda_i^{-1}\left(E_2(\tau)\Delta(\tau)^n A_n^{(1)}(j(\tau)) - E_4(\tau)^2 E_6(\tau)\Delta(\tau)^{n-1} B_n^{(1)}(j(\tau))\right) = q^{2n} + O(q^{2n+1})$$

Extremal quasimodular forms

$$f''(\tau) - \left(\frac{k}{6}E_2(\tau) - \frac{1}{3}\frac{E_6(\tau)}{E_4(\tau)}\right)f'(\tau) + \left(\frac{k(k-1)}{12}E_2'(\tau) - \frac{k-1}{18}\frac{E_6'(\tau)}{E_4(\tau)}\right)f(\tau) = 0$$

3. The Atkin orthogonal polynomials

Kaneko-Koike

$$\frac{E_2(j)\Delta(j)^n A_n^{(1)}(j) - E_4(j)^2 E_6(j)\Delta^{n-1}(j) B_n^{(1)}(j)}{E_4(j)^2 E_6(j)\Delta(j)^n A_n^{(1)}(j)} = O(j^{-2n-1})$$

$$\lambda_i = \begin{cases} 1 & n = 0 \\ 720 & n = 1 \\ 12\left(6 + \frac{(-1)^n}{n-1}\right)\left(6 + \frac{(-1)^n}{n}\right) & n \geq 2 \end{cases}$$

$$\prod_{i=0}^{2n} \lambda_i^{-1} (E_2(\tau)\Delta(\tau)^n A_n^{(1)}(j(\tau)) - E_4(\tau)^2 E_6(\tau)\Delta(\tau)^{n-1} B_n^{(1)}(j(\tau))) = q^{2n} + O(q^{2n+1})$$

Extremal quasimodular forms

$$f''(\tau) - \left(\frac{k}{6}E_2(\tau) - \frac{1}{3}\frac{E_6(\tau)}{E_4(\tau)}\right)f'(\tau) + \left(\frac{k(k-1)}{12}E_2'(\tau) - \frac{k-1}{18}\frac{E_6'(\tau)}{E_4(\tau)}\right)f(\tau) = 0$$

3. The Atkin orthogonal polynomials

Groups	Related to $ss_p(j)$	ex-qmf.
$\mathrm{SL}_2(\mathbb{Z})$	Kaneko-Zagier	Kaneko-Koike
$\Gamma_0(N)\ (N \leq 4)$	Tsutsumi	Tsutsumi-S. NO
$\Gamma_0^*(N)\ (N = 2, 3)$	?	?

$$\Gamma_0^*(N) = \Gamma_0(N) \cup \Gamma_0(N) \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$$

3. The Atkin orthogonal polynomials

$E_{k,N}(\tau) := \dfrac{N^{k/2}E_k(N\tau) + E_k(\tau)}{N^{k/2}+1}$: Eisenstein series of weight k for $\Gamma_0^*(N)$

$\Delta_{2A}(\tau) = \eta(2\tau)^8\eta(\tau)^8$: cusp form of weight 8 for $\Gamma_0^*(2)$

$\Delta_{3A}(\tau)^2 = \eta(3\tau)^{12}\eta(\tau)^{12}$: cusp form of weight 12 for $\Gamma_0^*(3)$

$\Delta_{3A}^{8+}(\tau) = I_3(\tau)^2\Delta_{3A}(\tau)$: cusp form of weight 8 for $\Gamma_0^*(3)$

$\Delta_{3A}^{10+}(\tau) = I_3(\tau)(I_3(\tau)^3 - 54\Delta_3(\tau))\Delta_{3A}(\tau)$: cusp form of weight 10 for $\Gamma_0^*(3)$

$M_*(\Gamma_0^*(2)) = \mathbb{C}[E_{4,2}, \Delta_{2A}] \bigoplus E_{6,2}\mathbb{C}[E_{4,2}, \Delta_{2A}]$ $\qquad M_*(\Gamma_0^*(3)) = \mathbb{C}[E_{4,3}, E_{6,3}, \Delta_{3A}^{8+}, \Delta_{3A}^{10+}, \Delta_{3A}^2]/\sim$

$$j_2^*(\tau) = \frac{E_{4,2}(\tau)^2}{\Delta_{2A}(\tau)}, \quad j_3^*(\tau) = \frac{I_3(\tau)^6}{\Delta_{3A}(\tau)}$$

3. The Atkin orthogonal polynomials

For $f,\ g \in \mathbb{C}[j_N^*]$, $\quad (f,g)_{\Gamma_0^*(N)} :=$ constant term of $f g E_{2,N}$ as a Laurent series in q

⇓ : Non-degenerate

$$A_{n+1}^{(2)}(j_2^*) = \left(j_2^* - 8\frac{(64n^2-11)}{(2n-1)(2n+1)}\right)A_n^{(2)}(j_2^*) - 4\frac{(8n-9)(8n-5)(8n-3)(8n+1)}{n(n-1)(2n-1)^2}A_{n-1}^{(2)}(j_2^*)$$

$$A_0^{(2)}(j_2^*) = 1,\ A_1^{(2)}(j_2^*) = j_2^* - 96 \text{ and } A_2^{(2)}(j_2^*) = {j_2^*}^2 - \frac{712}{3}j_2^* + 4928,$$

$$A_{n+1}^{(3)}(j_3^*) = \left(j_3^* - 6\frac{(36n^2-5)}{(2n-1)(2n+1)}\right)A_n^{(3)}(j_3^*) - 9\frac{(6n-7)(3n-2)(3n-1)(6n+1)}{n(n-1)(2n-1)^2}A_{n-1}^{(3)}(j_3^*)$$

$$A_0^{(3)}(j_3^*) = 1,\ A_1^{(3)}(j_3^*) = j_2^* - 36 \text{ and } A_2^{(3)}(j_3^*) = {j_3^*}^2 - 98j_3^* + 720,$$

3. The Atkin orthogonal polynomials

For $f,\ g \in \mathbb{C}[j_N^*]$, $\quad (f,g)_{\Gamma_0^*(N)} :=$ constant term of $fgE_{2,N}$ as a Laurent series in q

: Non-degenerate

$$A_n^{(2)}(j_2^*) = \sum_{i=0}^{n}(-256)^i\left[\sum_{m=0}^{i}\binom{-\frac{1}{8}}{i-m}\binom{-\frac{3}{8}}{i-m}\binom{n+\frac{1}{8}}{m}\binom{n-\frac{5}{8}}{m}\binom{2n-1}{m}^{-1}\right]{j_2^*}^{n-i}.$$

$$A_n^{(3)}(j_3^*) = \sum_{i=0}^{n}(-108)^i\left[\sum_{m=0}^{i}\binom{-\frac{1}{6}}{i-m}\binom{-\frac{1}{3}}{i-m}\binom{n+\frac{1}{6}}{m}\binom{n-\frac{2}{3}}{m}\binom{2n-1}{m}^{-1}\right]{j_3^*}^{n-i}.$$

3. The Atkin orthogonal polynomials

For a prime number $p(\geq 5)$, "supersingular j_N^*-polynomials for $\Gamma_0^*(N)$" by

$$S_p^{(2A)}(X) := X^{\delta_2}(X-256)^{\varepsilon_2}\begin{cases} X^{m_2}\mathbb{F}\left(\frac{1}{8},\frac{3}{8},1,\frac{256}{X}\right) & p \equiv 1,\ 3 \pmod 8, \\ X^{m_2}\mathbb{F}\left(\frac{7}{8},\frac{5}{8},1,\frac{256}{X}\right) & p \equiv 5,\ 7 \pmod 8, \end{cases}$$

$$S_p^{(3A)}(X) := X^{\delta_3}(X-108)^{\varepsilon_3}\begin{cases} X^{m_3}\mathbb{F}\left(\frac{1}{6},\frac{1}{3},1,\frac{108}{X}\right) & p \equiv 1 \pmod 6, \\ X^{m_3}\mathbb{F}\left(\frac{2}{3},\frac{5}{6},1,\frac{108}{X}\right) & p \equiv 5 \pmod 6, \end{cases}$$

$m_2 = [\frac{p}{8}],\ p-1 = 8m_2 + 2\delta_2 + 4\varepsilon_2,\ m_3 = [\frac{p}{6}],\ p-1 = 6m_3 + 2\delta_3 + 2\varepsilon_3,$
$\delta_N, \varepsilon_N \in \{0,\ 1\}$

These are induced from 'invariant differentials'

$$A_{n_p}^{(N)}(j_N^*) \equiv S_p^{(NA)}(j_N^*) \pmod p \qquad n_p = m_N + \delta_N + \varepsilon_N$$

3. The Atkin orthogonal polynomials

Y. Ihara

$$\left(\frac{dq}{q}\right)^{\frac{p-1}{2}} \equiv \frac{ss_p(j)}{j^{[\frac{p+1}{3}]}(j-1728)^{[\frac{p+1}{4}]}}dj^{\frac{p-1}{2}} \pmod p \qquad (p \geq 5)$$

Kaneko-Zagier

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

$k = p-1$ $\qquad 1 \equiv E_{p-1}(\tau) \equiv F_{p-1}(\tau) \pmod p$ $\qquad \widetilde{E}_{p-1}(j) \equiv \widetilde{F}_{p-1}(j) \pmod p$

$$F_{p-1}\left(\frac{dq}{q}\right)^{\frac{p-1}{2}} = g(j)dj^{\frac{p-1}{2}} \implies \left(\frac{dq}{q}\right)^{\frac{p-1}{2}} \equiv g(j)dj^{\frac{p-1}{2}}, \quad g(j) \equiv \frac{ss_p(j)}{j^{[\frac{p+1}{3}]}(j-1728)^{[\frac{p+1}{4}]}} \pmod p$$

3. The Atkin orthogonal polynomials

Koike-Saijo

$$f''(\tau) - \frac{k+1}{4} E_{2,2}(\tau) f'(\tau) + \frac{k(k+1)}{8} E'_{2,2}(\tau) f(\tau) = 0$$

$k = n(p-1)$ $\quad 1 \equiv E_{n(p-1),2}(\tau) \equiv F^{(2)}_{p-1}(\tau)^n \pmod{p}$ $\quad \widetilde{E}_{n(p-1),2}(j_2^*) \equiv \widetilde{F^{(2)}}_{p-1}(j_2^*)^n \pmod{p}$

$(n = 1 \text{ or } 2)$

$$F^{(2)}_{p-1}{}^2 \left(\frac{dq}{q}\right)^{p-1} = g_2(j_2^*) dj_2^{*\,p-1} \quad \Longrightarrow \quad \left(\frac{dq}{q}\right)^{p-1} \equiv g_2(j_2^*) dj_2^{*\,p-1}, \quad g_2(j_2^*) \equiv \frac{(S_p^{(2A)}(j_2^*))^2}{j_2^{*\Box}(j_2^* - 256)^{\Box}} \pmod{p}$$

3. The Atkin orthogonal polynomials

Koike-Saijo-S.

$$f''(\tau) - \frac{k+1}{3}E_{2,3}(\tau)f'(\tau) + \frac{k(k+1)}{6}E'_{2,3}(\tau)f(\tau) = 0$$

$k = n(p-1)$ $\quad 1 \equiv E_{n(p-1),3}(\tau) \equiv F^{(3)}_{p-1}(\tau)^n \pmod{p}$ $\quad \widetilde{E}_{n(p-1),3}(j_3^*) \equiv \widetilde{F^{(3)}}_{p-1}(j_3^*)^n \pmod{p}$

$(n = 1 \text{ or } 2)$

$$F^{(3)}_{p-1}{}^2\left(\frac{dq}{q}\right)^{p-1} = g_3(j_3^*)dj_3^{*\,p-1} \quad \Longrightarrow \quad \left(\frac{dq}{q}\right)^{p-1} \equiv g_3(j_3^*)dj_3^{*\,p-1}, \quad g_3(j_3^*) \equiv \frac{(S_p^{(3A)}(j_3^*))^2}{j_3^{*\,\square}(j_3^* - 108)^{\square}} \pmod{p}$$

3. The Atkin orthogonal polynomials

Ihara, Koike, Kaneko-Zagier

$$\left(\frac{dq}{q}\right)^{\frac{p-1}{2}} = \frac{ss_p(j)}{j^{\left[\frac{p+1}{3}\right]}(j-1728)^{\left[\frac{p+1}{4}\right]}} dj^{\frac{p-1}{2}}$$

3. The Atkin orthogonal polynomials

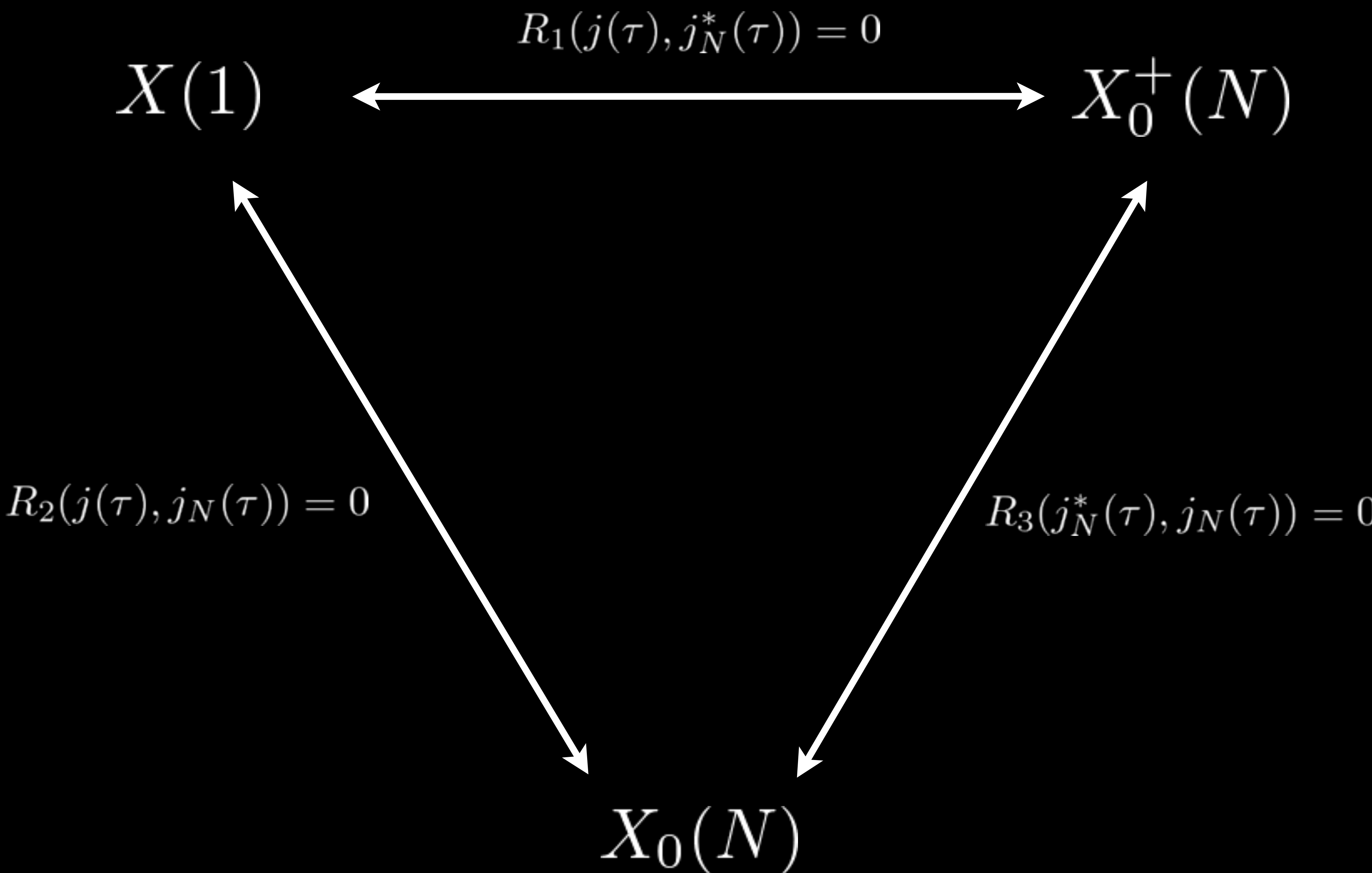
Ihara, Koike, Kaneko-Zagier

$$\left(\frac{dq}{q}\right)^{\frac{p-1}{2}} = \frac{ss_p(j)}{j^{[\frac{p+1}{3}]}(j-1728)^{[\frac{p+1}{4}]}} dj^{\frac{p-1}{2}}$$

$$\left(\frac{dq}{q}\right)^{p-1} = \frac{\left(S_p^{(2A)}(j_2^*)\right)^2}{{j_2^*}^{\square}(j_2^*-256)^{\square}} d{j_2^*}^{p-1}$$

$$\left(\frac{dq}{q}\right)^{p-1} = \frac{\left(S_p^{(3A)}(j_3^*)\right)^2}{{j_3^*}^{\square}(j_3^*-108)^{\square}} d{j_3^*}^{p-1}$$

3. The Atkin orthogonal polynomials



(Defined by Tsutsumi)

3. The Atkin orthogonal polynomials

$$\Phi^{(N)}(j_N^*) := \frac{E_{2,N}(j_N^*)E_{4,N}(j_N^*)}{E_{6,N}(j_N^*)j_N^*} = \frac{B_n^{(N)}(j_N^*)}{A_n^{(N)}(j_N^*)} + O({j_N^*}^{-2n-1})$$

$$\prod_{i=1}^{2n}(\lambda_i^{(2)})^{-1}\Big(E_{2,2}(\tau)\Delta_{2A}(\tau)^n A_n^{(2)}(j_2^*(\tau)) - E_{4,2}(\tau)E_{6,2}(\tau)\Delta_{2A}(\tau)^{n-1}B_n^{(2)}(j_2^*(\tau))\Big) = q^{2n} + O(q^{2n+1})$$

$$\lambda_n^{(2)} = \begin{cases} 96 & n=1 \\ 4\Big(4+\frac{(-1)^n}{n-1}\Big)\Big(4+\frac{(-1)^n}{n}\Big) & n>1 \end{cases}$$

$$f'' - \Big(\frac{k}{4}E_{2,2} - \frac{1}{2}\frac{E_{6,2}}{E_{4,2}}\Big)f' + \Big(\frac{k(k-1)}{8}E'_{2,2} - \frac{k-1}{12}\frac{E'_{6,2}}{E_{4,2}} - \frac{16}{3}(k-1)\frac{\Delta_{2A}}{E_{4,2}}\Big)f = 0.$$

3. The Atkin orthogonal polynomials

$$\Phi^{(N)}(j_N^*) := \frac{E_{2,N}(j_N^*)E_{4,N}(j_N^*)}{E_{6,N}(j_N^*)j_N^*} = \frac{B_n^{(N)}(j_N^*)}{A_n^{(N)}(j_N^*)} + O({j_N^*}^{-2n-1})$$

$$\prod_{i=1}^{4n}(\lambda_n^{(3)})^{-1}\Big(E_{2,3}(\tau)\Delta_{3A}(\tau)^{2n}A_{2n}^{(3)}(j_3^*(\tau)) - E_{4,3}(\tau)\Delta_{3A}^{10+}(\tau)\Delta_{3A}(\tau)^{2n-2}B_{2n}^{(3)}(j_3^*(\tau))\Big) = q^{4n} + O(q^{4n+1})$$

$$\lambda_n^{(3)} = \begin{cases} 36 & n = 1 \\ 3\left(3 + \frac{(-1)^n}{n-1}\right)\left(3 + \frac{(-1)^n}{n}\right) & n > 1 \end{cases}$$

$$f'' - \Big(\frac{k}{3}E_{2,3} - \frac{2}{3}\frac{E_{6,3}}{E_{4,3}}\Big)f' + \Big(\frac{k(k-1)}{6}E'_{2,3} - \frac{k-1}{9}\frac{E'_{6,3}}{E_{4,3}} + 6(k-1)\frac{\Delta_{3A}^{8+}}{E_{4,3}}\Big)f = 0.$$

3. The Atkin orthogonal polynomials

Questions

What type's polynomials can produce normalized ex-qmf?

Do there exist the system of quasimodular forms satisfying MDE with simple poles at ell. pt. ?

Do there exist orthogonal polynomials except for arithmetic triangle groups?

Thank you