Serre's Local Class Field Theory and Ramification Theory (joint w/ M. Yoshida)

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§1.

Th. (S.)

K : CDVF w/ alg. closed res. fld. k

L/K : fin. ab. wild.

$u_{L/K}$: the max. jump of $Gal(L/K)$ $\left(\text{i.e. } Gal(L/K)^{u_{L/K}} \neq 1,\ Gal(L/K)^{u_{L/K}+\varepsilon} = 1\right)$

$\longrightarrow$ (P_m) holds for $m = u_{L/K}$ i.e.

$\forall E/K$: alg. $\forall \varphi : O_L \longrightarrow O_E/\mathfrak{a}^m_{E/K}$: O_K-hom.

$\mathfrak{a}^m_{E/K} = \{x \in E \mid v_K(x) \geq m\}$

$\exists L \hookrightarrow E$: K-emb. □

Conj.

K : CDVF w/ perfect res. fld. k

L/K : fin. ab. wild.

Then,

(P_m) holds for $m = u_{L/K}$ $\Longleftrightarrow$ $\forall k'/k$: fin. Gal. $\mathrm{char}(k) \nmid [k':k]$ □

Rmk.

Conj. true if
- k : finite (Yoshida : using LCFT!)
- $\mathrm{char}(k) = p > 0$, $[L:K] = p$

  $u_{L/K} = 2$ (S.)

§2

$k = \bar{k}$ — ab. cat. & enough proj.

(The category of comm. pro alg. gp./k) $\longrightarrow$ (The category of profin. ab. gp.) : right exact

$G \longmapsto \pi_0(G) = G/G^\circ$ (G°: conn. comp.)

$L_i\pi_0(G) =: \pi_i(G)$ i-th homotopy gp.

K : CDVF w/ res. fld. $k = \bar{k}$

U_K : gp. of units

$\underline{U}_K = U_K$ w/ a proalg. gp. str. $/k$

$$\underline{U}_K(R) = \begin{cases} (W(R) \otimes_{W(k)} \mathcal{O}_K)^\times & \text{if } \mathrm{char}(K) \neq \mathrm{char}(k) \quad ① \\ (R \hat{\otimes}_k \mathcal{O}_K)^\times & \text{if } \quad " = " \quad ② \end{cases}$$

(R : k-alg.)

$$0 \to \underline{U}_K^n \subset \underline{U}_K \to \underline{U}_K/\underline{U}_K^n \to 0 \qquad \underline{U}_K = \varprojlim_n \underline{U}_K/\underline{U}_K^n$$

($\underline{U}_K/\underline{U}_K^n$: n-dim. alg. gp.)

① $\mathcal{O}_K = W(k)[x]/(f(x))$ $\quad f$: deg e

$$\underline{U}_K(R) = W(R)[x]/(f(x)) = W(R)^\times \oplus \bigoplus_{i=1}^{e-1} W(R)\bar{x}^i$$

$\therefore \underline{U}_K \cong W^\times \times W^{e-1}$ as a scheme

$\cong \mathbb{G}_m \times \mathbb{A}^{\mathbb{N}} \times (\mathbb{A}^{\mathbb{N}})^{e-1}$

Th. (Serre)

$\exists$ a canonical isom. : $\pi_1(\underline{U}_K) \xrightarrow{\cong} G_K^{ab}$

$$\begin{array}{ccccccccc} 0 \to & \pi_1(\underline{U}_K^m) & \to & \pi_1(\underline{U}_K) & \to & \pi_1(\underline{U}_K/\underline{U}_K^m) & \to 0 \\ & \downarrow ? & & \downarrow ? & & \downarrow ? & \\ 0 \to & (G_K^{ab})^m & \to & G_K^{ab} & \to & G_K^{ab}/(G_K^{ab})^m & \to 0 \end{array}$$ □

· Construction :

L/K : fin. ab. $\qquad \underline{\pi_L} = 1 \otimes \pi_L \in W(R) \otimes_{W(k)} \mathcal{O}_L$

$$0 \to G(L/K) \to \underline{U}_L/\underline{V}_{L/K} \xrightarrow{N_{L/K}} \underline{U}_K \to 0$$

$$\sigma \longmapsto \underline{\pi_L}^{\sigma - 1}$$

$\underline{V}_{L/K} = \pi_0(\mathrm{Ker}\, N_{L/K}) = \langle x^{\sigma-1} \mid x \in \underline{U}_L,\ \sigma \in G(L/K) \rangle$

$\rightsquigarrow \pi_1(\underline{U}_K) \xrightarrow{\partial} \pi_0(G(L/K)) = G(L/K) \to \pi_0(\underline{U}_L/\underline{V}_{L/K}) = 1$

$\xrightarrow{\varprojlim_{L/K}} \pi_1(\underline{U}_K) \to G_K^{ab}$ (isom.)

· $\forall m \geq 1$

$$0 \to G(L/K)/G(L/K)^m \to \underline{U}_L/\underline{U}_L^{\psi_{L/K}(m-1)+1}\underline{V}_{L/K} \xrightarrow{N_{L/K}} \underline{U}_K/\underline{U}_K^m \to 0$$

· E/K : fin.

$\Longrightarrow 0 \to \mathrm{Ker}(N_{E/K}) \to \underline{U}_E \xrightarrow{N_{E/K}} \underline{U}_K \to 0$

$\pi_0(\mathrm{Ker}(N_{E/K})) \cong G(E \cap K^{ab}/K)$

§3 Proof

$k = \bar{k}$, L/K : fin. ab. wild. $\quad (u_{L/K} \in \mathbb{Z}_{>1})$ Hasse-Arf

Assume we are given

$\varphi : \mathcal{O}_L \to \mathcal{O}_E/\mathfrak{m}_E^m$: $\mathcal{O}_K$-hom.

$m \in \mathbb{Z}_{\geq 1} \rightsquigarrow$

$$\begin{array}{ccc} \mathcal{O}_L/\mathfrak{m}_K^m & \underset{\text{fin. free}}{\overset{\textcircled{\varphi}}{\hookrightarrow}} & \mathcal{O}_E/\mathfrak{m}_K^m \\ \text{fin. free} \uparrow & \circlearrowleft & \uparrow \text{ fin. free} \\ \mathcal{O}_K/\mathfrak{m}_K^m & = & \mathcal{O}_K/\mathfrak{m}_K^m \end{array}$$

Take the norm :

$$\begin{array}{ccccc} & U_L/U_L^{m e_{L/K}} & \xleftarrow{\textcircled{N_\varphi}} & U_E/U_E^{m e_{E/K}} & \xleftarrow{\text{can.}} U_E \\ U_L/U_L^{\psi_{L/K}(m-1)+1} \xleftarrow{\text{can.}} & \downarrow N_{L/K} & \circlearrowleft & \downarrow N_{E/K} & \\ N_{L/K} \searrow & U_K/U_K^m & = & U_K/U_K^m & \end{array}$$

$\psi_{L/K}(m-1)+1 \leq m e_{L/K}$

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}(N_{E/K}) & \to & U_E & \xrightarrow{N_{E/K}} & U_K & \to & 0 \\ & & \downarrow \psi & & \downarrow N_\varphi & & \downarrow \text{can.} & & \\ 0 & \to & G(L/K)/G(L/K)^m & \to & U_L/U_L^{\psi_{L/K}(m-1)+1} & \xrightarrow{N_{L/K}} & U_K/U_K^m & \to & 0 \end{array}$$

$G(L/K)^m = 1$ — $m \gg u_{L/K}$

Take π_i $\rightsquigarrow$

$$\begin{array}{ccccc} G_K^{ab} \cong \pi_1(U_K) & \xrightarrow{\partial} & \pi_0(\mathrm{Ker}(N_{E/K})) = G(E\cap K^{ab}/K) & & (\text{res.}: G_K^{ab} \to G(E\cap K^{ab}/K)) \\ \downarrow \text{can.} & \circlearrowleft & \swarrow \psi & & \\ G_K^{ab}/(G_K^{ab})^m \cong \pi_1(U_K/U_K^m) & \xrightarrow{\partial} & G(L/K) & & (\text{res.}) \end{array}$$

$$\begin{array}{ccc} G_K^{ab} & \overset{\text{res.}}{\twoheadrightarrow} & G(E\cap K^{ab}/K) \\ & \underset{\text{res.}}{\searrow} \circlearrowleft & \downarrow \psi \\ & & G(L/K) \end{array} \rightsquigarrow L \subset E \quad //$$

Compliments

$\pi_0(G)^\vee := \mathrm{Hom}(\pi_0(G), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$

$\therefore \pi_i(G)^\vee = \mathrm{Ext}^i(G, \mathbb{Q}/\mathbb{Z})$

$\mathrm{Hom}(\pi_1(G), \underset{\sim}{N}) = \mathrm{Ext}^1(G, N) \leftrightarrow (0 \to N \to \square \to G \to 0)$

N: fin. ab.