Sato theory, *p*-adic tau function and arithmetic geometry

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References

This talk is mainly devoted to a survey of the paper

[A] Anderson, *Torsion points on Jacobians of quotients of Fermat curves and p-adic soliton theory*, Invent. Math. 118 (1994), 475–492

in which the ***p***-adic analogue of the Sato theory is developed and applied to arithmetic geometry.

A basic reference to the Sato theory is

[SW] Segal, Wilson, *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math. 61 (1985), 5–65.

Sato theory

Mikio Sato's theory on 'soliton' nonlinear PDEs, such as the Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0$$

($u = u(t,x)$, a model for waves on shallow water surfaces) and the Kadomtsev-Petviashvili (KP) equation

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}\right) + \frac{\partial^2 u}{\partial y^2} = 0$$

($u = u(t,x,y)$, two-dimensional KdV equation), has a strong connection with algebraic curves.

Elliptic and theta solutions

Look for a solution to KdV $u_{xxx} = 6uu_x + u_t$ of the form $u = f(x - ct)$ (f : one-variable function, c : constant).

- $f''' = 6ff' - cf'$
- $f'' = 3f^2 - cf + b$ (b : constant)
- $f''f' = (3f^2 - cf + b)f'$
- $\frac{1}{2}(f')^2 = f^3 - \frac{c}{2}f^2 + bf + a$ (a : constant)

Conclusion : The Weierstrass $\wp$ function of an elliptic curve $\frac{1}{2}y^2 = x^3 - \frac{c}{2}x^2 + bx + a$ gives rise to a solution to KdV.

More generally, the theta function of each hyperelliptic curve gives rise to a solution to the KdV equation (hierarchy).
The theta function of each algebraic curve gives rise to a solution to the KP equation (hierarchy).

Manin-Mumford conjecture

C : proper smooth curve over $\mathbb{C}$
J : Jacobian variety, J_{Tor} : torsion subgroup
Θ : theta divisor on J (zero locus of the theta function)

Manin-Mumford conjecture (Raynaud's theorem) implies:

$$J_{\mathrm{Tor}} \cap \Theta \text{ is a finite set}$$

Problem : Determine this finite set explicitly.

Quite a lot of works have been done, especially in the cases of modular curves, Fermat curves and their quotients.

Example (Coleman, Kaskel, Ribet (1999)): For $C = X_0(37)$,

$$J_{\mathrm{Tor}} \cap \Theta = \{ \text{ two cusps } \}$$

Anderson's result

$C : y^l = x^a(1-x)^{l+1-a}$, l : odd prime, $0 < a < l$
(cyclic quotient of the Fermat curve of degree l)
$\mathbb{Z}[\zeta_l]$ acts on C (hence on J) by $[\zeta_l](x, y) = (x, \zeta_l y)$
$\mathfrak{p} \subset \mathbb{Z}[\zeta_l]$: prime ideal, $\mathfrak{p} \mid (p)$, p : prime, $p \equiv 1 \mod l$

Theorem. $J[(1-\zeta_l)\mathfrak{p}] \cap \Theta = J[(1-\zeta_l)] \cap \Theta$

Excerpt from [A] : *The proof of [the above theorem] is only the secondary purpose of this paper. The primary one is to initiate diophantine applications of soliton theory...*

Strategy

Given $\boldsymbol{P} \in \boldsymbol{J}_{\mathbf{Tor}}$, one wants to detect $\boldsymbol{P} \notin \boldsymbol{\Theta}$.
Recall that $\boldsymbol{\Theta}$ is the zero locus of the theta function θ.
We introduce the tau function with properties:

- a strong method (Sato expansion theorem) of computation at one's disposal;
- the zero locus of the tau function 'coincides' with $\boldsymbol{\Theta}$;
- however the tau function is not defined on $\boldsymbol{J}$; it is a function on the (infinite dimensional) loop group acting on the (again infinite dimensional) Sato Grassmannian

The rest of this talk is (mostly) devoted to a construction of the tau function with the above properties.

Sato Grassmannian

k : field; $H = k((\frac{1}{t}))$, $H_+ = tk[t]$, $H_- = k[[\frac{1}{t}]]$
Definition. The Sato Grassmannian Gr is the set of linear subspaces V of $k((\frac{1}{t}))$ such that the kernel and cokernel of

$$f_V : V \hookrightarrow H \to H/H_-(\cong H_+)$$

are finite dimensional; we further set for each $i \in \mathbb{Z}$
$Gr^i = \{V \in Gr \mid \dim \ker(f_V) - \dim \operatorname{coker}(f_V) = i\}$.

Remark. $V \subset H$ belongs to Gr^i iff $\exists w : H_+ \hookrightarrow H = H_+ \oplus H_-$
$V = w(H_+)$, $t^i w = (1 + u, v)$, u : finite rank

Basic example : $t^i k[t]$ is in Gr^{1-i}
This corresponds to the case of $C = \mathbb{P}^1$ in the following:

Krichever pair

C/k : smooth projective irreducible curve of genus g
Fix $\infty \in C(k)$ and an isomorphism $N_0 : k[[\frac{1}{t}]] \cong \hat{\mathcal{O}}_{C,\infty}$

Roughly speaking,

- $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$, coordinated by N_0, is in Gr^{1-g}
- $\mathcal{L} \in \mathrm{Pic}(C)$, σ : trivialization of $\mathcal{L}$ at ∞, $\deg(\mathcal{L}) = n$,
  $L = \Gamma(C \setminus \{\infty\}, \mathcal{L})$, coordinated by σ, is in Gr^{n+1-g}

Remark : Such a pair $(\mathcal{L}, \sigma)$ is called a Krichever pair.
L belongs to $Gr_A := \{V \in Gr \mid AV \subset V\}$.

Precisely speaking, $N : \mathrm{Spec}\, H \to C$: induced by N_0

- $A = \{N^* s \in H \mid s \in \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$
- $\sigma = \sigma_0 \otimes H$, $\sigma_0 : N_0^* \mathcal{L} \cong k[[\frac{1}{t}]]$,
  $L = \{\sigma N^* s \in H \mid s \in \Gamma(C \setminus \{\infty\}, \mathcal{L})\}$

Krichever correspondence

Krichever showed that this construction defines bijections

$$\begin{array}{ccccccc}
\{(\mathcal{L},\sigma)\}/\cong & \twoheadrightarrow & \mathrm{Pic}(C) & \supset & J & \supset & \Theta \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
Gr_A & & \twoheadrightarrow (Gr_A/\equiv) & & \supset (Gr_A^{1-g}/\equiv) & & \supset (X/\equiv)
\end{array}$$

$(\mathcal{L},\sigma) \cong (\mathcal{L}',\sigma') \Leftrightarrow \mathcal{L} \cong \mathcal{L}$ compatible with σ,σ'
$V \equiv V' \Leftrightarrow V' = uV$ for some $u \in k[[\frac{1}{t}]]^* \quad (V, V' \in Gr)$
$Gr_A^i = Gr^i \cap Gr_A$
$X = \{V \in Gr_A^{1-g} \mid V \cap t^{g-1}k[[\frac{1}{t}]] \neq 0\}$

The last correspondence follows from

- $\Theta = \{\mathcal{L} \in J \mid H^0(C, \mathcal{L}((g-1)\infty) \neq 0\}$,
- $H^0(C, \mathcal{L}((g-1)\infty)) = \ker(L \to k((\frac{1}{t}))/t^{g-1}k[[\frac{1}{t}]])$.

Loop group

To proceed further, one has to consider analytic version, but in this talk we pretend as if no ‘convergent problem’ exists. Let (p : prime and $[k : \mathbb{Q}_p] < \infty$) or ($p = \infty$ and $k = \mathbb{C}$).

Definition. We define the loop group to be

$$\Gamma = \{1 + \sum_{n=1}^{\infty} a_n t^n \in k[[t]]^*\};$$

for $h \in \Gamma$ and $w \in H$, one ‘defines’ a product $hw \in H$, which induces an action of a big group Γ on H and on Gr.

Actually, this action is not well-defined, since a product hw of $h \in \Gamma$ and $w \in H = k((\frac{1}{t}))$ cannot be well-defined; this problem is resolved by introducing analytic version.

Tau function

Suppose $W \in Gr$ is given by $W = w(H_+)$ with
$t^i w = (1 + u, v) : H_+ \hookrightarrow H = H_+ \oplus H_-$, u: finite rank.
We define the tau function $\tau_W : \Gamma \to k$ by

$$\tau_W(h) = \det(H_+ \xrightarrow{t^i w} H \xrightarrow{h} H \xrightarrow{\text{proj}} H_+).$$

To define the determinant of an endomorphism on an infinite dimensional space H_+, one again has to use analytic theory. (p-adic case : Serre's theory of p-adic Banach space.)

Key Lemma. Take $W \in Gr_A^{1-g}$. For $h \in \Gamma$, $\tau_W(h) = 0$ iff hW falls in Θ via the Krivever correspondence.
Proof. $\tau_W(h) = 0 \Leftrightarrow \ker(\ldots) \neq 0 \Leftrightarrow t^{1-g}W \cap h^{-1}H_- \neq 0$
$\Leftrightarrow hW \cap t^{g-1}H_- = hW \cap t^{g-1}k[[\frac{1}{t}]] \neq 0 \Leftrightarrow hW \in X \twoheadrightarrow \Theta$.

Tau and theta (detour)

τ and θ share Θ as their zero loci, suggesting the following
Theorem. (cf. [SW §9]) When $k = \mathbb{C}$, we have

$$\tau_W(h(\vec{x})) = (\text{linear exponential factor})\theta(\vec{x}).$$

Here both sides are functions of $\vec{x} = (x_1, x_2, \cdots, x_g)$ via

- $h(\vec{x}) := \exp(x_1 t + x_2 t^2 + \cdots + x_g t^g)$ $(\in \Gamma)$,
- θ is a function on a universal covering $\mathbb{C}^g \twoheadrightarrow J$.

(A lot of choice and complicated normalization required.)
$u := (\log \tau_W \circ h)_{x_1 x_1} = (\log \theta)_{x_1 x_1}$ satisfies the KP equation upon substituting $x_1 = t + x, x_2 = y$, as well as a family of PDEs, called the KP hierarchy, involving u_{x_i} for $i > 2$.

Problem. Is there a p-adic analogue for the Tate/Mumford curves or curves with good ordinary reduction?

Sato expansion theorem

The tau function admits the following Sato expansion

$$\tau_W(h) = \sum_\lambda P_\lambda(W) S_\lambda(h)$$

- $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ runs all partitions $(\lambda_i, r \in \mathbb{N})$
- $P_\lambda(W) =$ the Plücker coordinate of W at λ
  $= \det(w_{i,j-\lambda_i-i})_{ij}$ for a good basis $\{w_i = \sum w_{ij} t^j\}_i$ of W
- $S_\lambda(h) =$ the Schur polynomial of λ
  $= \det(h_{\lambda_i-i+j})_{ij}$ for $h = \sum_i h_i t^i$

Proof. Apply the Laplace expansion of (infinite) determinant to the definition $\tau_W(h) = \det(H_+ \xrightarrow{t^i w} H \xrightarrow{h} H \xrightarrow{\text{proj}} H_+)$.

End of the survey of [A]. Now we discuss the Sato theory.

Quick review on Plücker embedding

The Grassmannian $Gr(n, N) = \{W \subset k^N \mid \dim W = n\}$

admits the Plücker embedding $\left(0 \leq n \leq N,\ d = \binom{N}{n} - 1\right)$

$$Gr(n, N) \subset \mathbb{P}^d; \qquad W \mapsto (P_\mu(W) = \det(w_{i,\mu_j})_{ij})_\mu,$$

- $\mu = (1 \leq \mu_1 < \cdots < \mu_n \leq N)$
- $\{w_i = (w_{ij})_j\}_i$ is a basis of W;

by which $Gr(n, N)$ is identified with the closed subvariety of $\mathbb{P}^d$ defined by the Plücker bilinear relations.

Example. $Gr(2, 4) \subset \mathbb{P}^5$ is defined by

$$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0.$$

Plücker relation and KP hierarchy

The Schur functions $\{S_\lambda(h)\}_\lambda$ form an orthogonal basis of the Hilbert space $\mathcal{X}$ of functions on Γ; any $\xi \in \mathcal{X}$ is written as

$$\xi(h) = \sum_\lambda P_\lambda(\xi) S_\lambda(h).$$

Theorem (Sato). (1) For $\xi \in \mathcal{X}$; $\exists W \in Gr$ s.t. $\xi = \tau_W$ iff $\{P_\lambda(\xi)\}_\lambda$ satisfies the (infinite) Plücker relations.
(2) The Plücker relation is 'equivalent' to the KP hierarchy.
In other words,

- $Gr \subset \mathbb{P}(\mathcal{X})$ is defined by the Plücker relations,
- $Gr \subset \mathbb{P}(\mathcal{X})$ parametrize 'all' solutions to KP.

Combined with the relation of τ to θ, one gets solutions to KP arising from the theta functions of algebraic curves.

Other problems (1)

- (Repetition) Establish a formula connecting the p-adic tau and theta functions for the Tate/Mumford curves or curves with good ordinary reduction.
- Deduce 'p-adic theta solutions' to the KP hierarchy, using an answer to the above problem.
- Ichikawa proved this for Mumford curves without using the tau function.

Other problems (2)

- Fay's formula on the vanishing order of θ at $\boldsymbol{P}$ along $\vec{\boldsymbol{v}}$ ($\boldsymbol{P} \in \Theta$, $\vec{\boldsymbol{v}}$: tangent vector of $\boldsymbol{C}$ embedded in $\boldsymbol{J}$ at $\boldsymbol{P}$) is re-proved in [SW] using Sato expansion theorem. Birkenhake-Vanhaecke find the third geometric proof.
- Anderson claims [A] is a $\boldsymbol{p}$-adic analogue of this result (although $\boldsymbol{P}$ is not on Θ in this case). There is technical difficulty to generalize his method to other situation.
- Can one prove the $\boldsymbol{p}$-adic analogue of Fay's formula in the style of Birkenhake-Vanhaecke, hopefully for more general situation (general $\boldsymbol{C}$ and $\boldsymbol{P}$)?

Other problems (3)

- When one considers a positive characteristic version, it seems that the Drinfeld module naturally comes up.
- The l-adic Sato theory (with $l \neq p$ or $l = \infty$) could be useful to deal with p-torsion point.
- For a fixed $W \in Gr_A^{1-g}$, give a description of the image of

$$\Gamma \to Gr_A^{1-g} \twoheadrightarrow J; \quad h \mapsto [hW]$$

in terms of the Néron model/formal group.