A refinement of Krasner's lemma and ramification theory

(joint work with T. Suzuki)

§0

K : CDVF, $\bar{k}$: perfect residue field, $\mathrm{char}(\bar{k}) = p > 0$

v_K : valuation on $\bar{K}$ s.t. $v_K(K^\times) = \mathbb{Z}$, O_K : val. ring

For M/K : alg. ext., O_M : int. clos. of O_K in M

· $f = \sum a_i X^{e-i}$, $g = \sum b_i X^{e-i} \in O_K[X]$: monic irreducible

<u>Problem</u> $\quad O_K[X]/f \;\cong\; O_K[X]/g \quad ?$

$\quad\; \overset{\shortparallel}{O_L} \qquad\qquad \underset{\text{f.s.}}{\overset{\shortparallel}{}} \qquad\qquad \overset{\shortparallel}{O_{M_g}}$

<u>e.g. n Thm.</u> $\quad e = p$ (for simplicity)

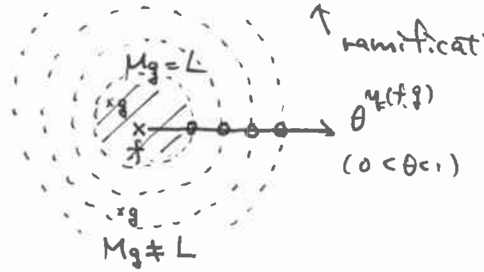
L/K : Gal. tot. ram. ext. (f.s.) $\qquad$ <u>fix</u>

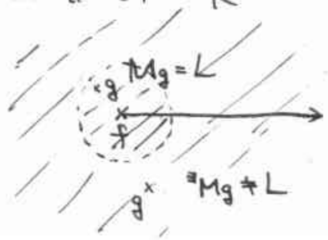
M_g/K : any ext. $\qquad$ <u>move</u>

$$v_K(f,g) := \min_i \left\{ v_K(a_i - b_i) + \frac{i}{p} \right\} \in \frac{1}{p}\mathbb{Z}$$

$$v_K(f,g) \begin{cases} > u_{L/K} & \longrightarrow \; L = M_g \\ \leq u_{L/K} \text{ and } \notin \mathbb{Z} & \longrightarrow \; L \neq M_g \end{cases}$$

$\uparrow$ ramification break of L/K





<u>Today</u> Study the behavior at the break $u_{L/K}$!

§1. Distance on polynomials

§2. Krasner's lemma and a refinement

§3. Proof of Thm. B

§4. To Suzuki

– free talk –

§1

· P_K : all monic irred. poly.

$\quad\cup$

· E_K^e : all Eisenstein poly. of $\deg = e$

· For $f, g \in P_K$, $v_K'(f,g) := v_K(\mathrm{Res}(f,g))$

resultant of f and g ($\mathrm{Res}(f,g) = 0 \iff f(\alpha) = g(\alpha) = 0$)

$\rightsquigarrow$ $v_K'(\ ,\ )$ define a non-Archimedean distance on P_K!

· If we restrict to E_K^e, then we use its normalized one:

For $f, g \in E_K^e$, $v_K(f,g) := \frac{1}{e} v_K'(f,g)$

✱ <u>Basic Settings</u>

$L = K(\alpha)$, $f(\alpha) = 0$, $O_L = O_K[\alpha]$

$M_g = K(\beta)$, $g(\beta) = 0$, $f, g \in P_K$

$\left(\begin{array}{ll} L/K : \text{Gal. ext.} & \text{fix} \\ M_g/K : \text{alg. ext.} & \text{move} \end{array} \right.$

□

§2

Notation $G := \mathrm{Gal}(L/K)$, e : ram. ind.

$G(i) := G_{ei-1}$, $i_{L/K} := \inf\{ i \in \mathbb{R} \mid G(i) = 1 \}$

$G(u) := G^{u-1}$, $u_{L/K} := \inf\{ u \in \mathbb{R} \mid G^{(u)} = 1 \}$ □

Krasner's lemma

$\alpha \in K^{sep}$, $\beta \in \overline{K}$, $\alpha = \alpha_1, \dots, \alpha_n$: conjugate

$v_K(\alpha - \beta) > \underbrace{\sup_{i \neq 1} v_K(\alpha - \alpha_i)}_{= i_{L/K}} \Longrightarrow K(\alpha) \subset K(\beta)$ □

under ※

· For $m \in \mathbb{R}_{\geq 0}$, $f \in P_K$

(K_m) $\forall g \in P_K$, $v_K'(f, g) \geq \deg(g)\, m \Longrightarrow L \underset{K}{\hookrightarrow} M_g$ □

Lem. A $m > u_{L/K} \Longrightarrow (K_m)$ true □

Thm. 1 (Fontaine, Y)

(K_m) true $\Longrightarrow m \geq u_{L/K}$ □

Thm. A L/K : abel. wild. $k = \bar{k}$

$\Longrightarrow (K_m)$ true for $m = u_{L/K}$ □

· For $m \in \mathbb{R}_{\geq 0}$, $f \in E_K^e$

(K_m') $\forall g \in E_K^e$, $v_K(f, g) \geq m \Longrightarrow L \simeq M_g$ □

Lem. B $m > u_{L/K} \Longrightarrow (K_m')$ true □

Thm. B $u_{L/K} \in \mathbb{Z}$, L/K : wild.

(K_m') for $m = u_{L/K} \Longleftrightarrow k = \wp(k)$ □

$\wp(X) := X^p - X$

§3

$f = X^e + a_{e-1} X^{e-1} + \cdots + a_1 X + a_0$ L/K fix

$g = X^e + b_{e-1} X^{e-1} + \cdots + b_1 X + b_0$ M_g/K move

Lem. 2 $v_K(f, g) = \min_i \{ v_K(a_i - b_i) + \frac{i}{e} \}$

Step 1 suffices to consider

$g_u := g = X^e + a_{e-1} X^{e-1} + \cdots + a_1 X + u a_0$ M_u/K

$u \in U_K^{u_{L/K}-1} \setminus U_K^{u_{L/K}}$

☺ Lem. 2, $u_{L/K} \in \mathbb{Z}$, Lem. B //

Step 2 「$\forall g_u .\ L = M_u$」 $\Leftrightarrow$ $N : U_L^{e i_{L/K}-1} / U_L^{e i_{L/K}} \to U_K^{u_{L/K}-1} / U_K^{u_{L/K}}$ is surj.

"$\Rightarrow$" $v_K(f, g_u) = u_{L/K}$) $\tilde{\varphi}$ ← Herbrand func. (Fontaine)

$\rightsquigarrow$ $v_K({}^{\exists}\pi_L - \pi_{M_u}) = i_{L/K}$

$\underset{L=M_u}{\rightsquigarrow}$ ${}^{\exists}u' \in U_L$ s.t. $\pi_{M_u} = u'\pi_L$, $N(u') = u$ //

$(u' \in U_L^{e i_{L/K}-1} \setminus U_L^{e i_{L/K}})$

"$\Leftarrow$" ${}^{\exists}u' \in$ " s.t. $N(u') = u$

$\pi_L' := u'\pi_L$

Take $f' \in E_K^e$ s.t. $f'(\pi_L') = 0$ $\quad f' : L/K$

$X^e + \cdots + u a_0$

$\rightsquigarrow$ $v_K(\pi_L - \pi_L') = i_{L/K}$) $\tilde{\varphi}^{-1}$

$\longrightarrow$ $v_K(f, f') = u_{L/K}$

$v_K(f, g_u) = u_{L/K}$) $\xrightarrow[\text{ultrametric}]{}$ $v_K(f', g_u) \geq u_{L/K}$

$\rightsquigarrow$ $v_K(f', g_u) > u_{L/K}$ $\xrightarrow[\text{Lem. B}]{}$ $L = M_u$ //

$\neq u_{L/K}$

$\because$ Lem. 2

Step 3 $\mathrm{Coker}(N) \cong \bigoplus \bar{k}/\wp(\bar{k})$

$n \in \mathbb{N}$

$$0 \to \overset{\text{カナイ}}{G_{\psi(n)}/G_{\psi(n)+1}} \to U_{L_0}^{\psi(n)}/U_{L_0}^{\psi(n)+1} \xrightarrow{N} U_{K_0}^{n}/U_{K_0}^{n+1} \to 0$$

$G_k \curvearrowright$ triv.

$$\begin{array}{ccc} \bar{k} & \longrightarrow & \bar{k} \\ G_k \curvearrowright & & G_k \curvearrowright \end{array}$$

$$\begin{array}{ccc} L & \xrightarrow{G_k} & L_0 := \hat{L}^{ur} \\ G \mid & & \mid G \\ K & \xrightarrow{G_k} & K_0 := \hat{K}^{ur} \end{array}$$

$\rightsquigarrow$ $\mathrm{Coker}(N) \cong \mathrm{Hom}(G_k, G_{\psi(n)}/G_{\psi(n)+1}) \cong \bigoplus \bar{k}/\wp(\bar{k})$ //

$\cong \bigoplus \mathbb{Z}/p$ $\qquad$ ↑ Artin-Schreier

§4

For $m \in \mathbb{R}$. L/K : fin. Gal. ext.

(P_m) $\forall E/K$: alg. ext.

$\mathcal{O}_L \xrightarrow[\mathcal{O}_K]{\exists} \mathcal{O}_E/\mathfrak{a}_{E/K}^m \Rightarrow L \overset{\exists}{\underset{K}{\hookrightarrow}} E$ $\square$

$\mathfrak{a}_{E/K}^m := \{x \in \mathcal{O}_E \mid v_K(x) \geq m\}$

$\rightsquigarrow$ $(P_m) \Leftrightarrow (K_m)$

$\rightsquigarrow$ Thm. A $\Leftrightarrow$ L/K : fin. ab. wild. $k = \bar{k}$

(P_m)

not true | true $\to m \in \mathbb{R}$

$u_{L/K}$