3-10 R-module N : fin. rep's

$\iff \exists m, n \in \mathbb{N}$ s.t. $R^n \longrightarrow R^m \longrightarrow N \longrightarrow 0$: exact

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$$0 \longrightarrow L \xrightarrow{\iota} M \xrightarrow{\pi} N \longrightarrow 0 \quad \text{exact.}$$

L, N : fin. rep. R-module $\Longrightarrow$ M : fin. rep.

$$R^{n_1} \xrightarrow{\psi_1} R^{m_1} \xrightarrow{\varphi_1} N \longrightarrow 0$$

$$R^{n_2} \xrightarrow{\psi_2} R^{m_2} \xrightarrow{\varphi_2} L \longrightarrow 0$$

R- hom. $\varphi : R^{m_1} \longrightarrow M$. exist?

$\varphi_1 = \pi \circ \varphi$

$$\begin{tikzcd} R^{m_1} \arrow[r, "\varphi"] \arrow[rr, bend right, "\varphi_1"'] & M \arrow[r, "\pi"] & N \end{tikzcd}$$

surj

surj.

$\mathrm{Im}(\underbrace{\varphi \circ \psi_1}_{R^{n_1} \to M}) \subset \underline{\mathrm{Ker}\, \pi = \mathrm{Im}\, \iota}$ exact 故.

$$\begin{tikzcd} R^{n_1} \arrow[r, "\psi_1"] & R^{m_1} \arrow[r, "\varphi"] \arrow[rr, bend right, "\varphi_1"'] & M \arrow[r, "\pi"] & N \end{tikzcd}$$

exact より $\mathrm{Im}(\varphi_1 \circ \psi_1) = 0$.

$\therefore \mathrm{Im}(\varphi \circ \psi_1) \subset \mathrm{Ker}\, \pi$ (でないと合成の行先が0でない)

$\Longrightarrow \exists \psi : R^{n_1} \longrightarrow R^{m_2}$ s.t. $\iota \circ \varphi_2 \circ \psi = \varphi \circ \psi_1$

$$\begin{tikzcd} R^{n_1} \arrow[r, "\psi_1"] \arrow[dr, "\psi"'] & R^{m_1} \arrow[drr, "\varphi"] & & \\ & R^{m_2} \arrow[r, "\varphi_2"'] & L \arrow[r, "\iota"'] & M \end{tikzcd}$$

$$\therefore \quad R^{n_1} \times R^{n_2} \xrightarrow{g} R^{m_1} \times R^{m_2} \xrightarrow{f} M \longrightarrow 0$$

$$(x, y) \longmapsto (\psi_1(x),\ \psi(x) - \psi_2(y))$$

$$(x, y) \longmapsto \varphi(x) - \iota \circ \varphi_2(y)$$

$$\operatorname{Im} g = \operatorname{Ker} f$$

$$\Longrightarrow \quad R^{n_1+n_2} \longrightarrow R^{m_1+m_2} \longrightarrow M \to 0 \quad : \underline{\underline{\text{exact.}}}$$

$\rightsquigarrow$ (Rq. <u>remark</u>)

$$\left(\begin{tikzcd}
R^{n_1} \arrow[r, "\psi_1"] \arrow[dr, "\psi"'] & R^{m_1} \arrow[drr, "\varphi"] & & \\
\times & \times & & \\
R^{n_2} \arrow[r, dashed] & R^{m_2} \arrow[r, "\varphi_2"'] & L \arrow[r, "\iota"'] & M
\end{tikzcd}\right)$$

exist ψ : $\operatorname{Im}(\varphi \circ \psi_1) \subset \operatorname{Ker} \pi = \operatorname{Im} \iota$

$\rightsquigarrow$ $\exists\, l \in L$ exists.

$\rightsquigarrow$ $R^{m_2} \xrightarrow{\varphi_2} L \to 0$: exact $\to$ φ_2: surject.

i.e. $\exists\, \underline{r_2 \in R^{m_2}}$

$\therefore$ R^{n_1} $\exists$ <u>corresp</u>.

$$\begin{tikzcd}
R^{m_1} \arrow[r, "\varphi_1\ Sj"] \arrow[d, "\exists\varphi"'] & N \\
M \arrow[ur, "\pi\ Sj'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R^{n_1} \arrow[r, "Sj'"] \arrow[d, "\exists\psi"'] & M \\
R^{m_2} \arrow[ur, "Sj"'] &
\end{tikzcd}$$

$\longleftarrow$ <u>exist R-hom. φ and ψ.</u>