MMA 数学特論 I。多項式系のアルゴリズム: グレブナー基底 & 消去法
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Practice test III: Around the Buchberger algorithm

- You can use any theorem, proposition or corollary of the class lectures, just by citing its number inside the corresponding lecture: (example: "Lect II, Cor. 1" refers to the Corollary 1 of Lecture II, that is the Primitive Element Theorem).

Exercise 1 Write the correct answer in the table below, and then compute the S-polynomials afterwards.

$$\begin{aligned} f(x,y,z) &= x^3z^5 - x^2yz^5 + 2xyz^6 \\ g(x,y,z) &= y^3 - y^2z + 3xy^2. \end{aligned}$$

$\prec$ is $\rightarrow$	$grlex(x,y,z)$	$grlex(z,x,y)$	$grevlex(y,x,z)$
$\mathrm{LT}_\prec(f)$			
$\mathrm{LT}_\prec(g)$			
$\mathrm{LCM}(\mathrm{LM}_\prec(f), \mathrm{LM}_\prec(g))$			

Computation of $S_\prec(f,g)$ for $\prec = \prec_{grlex(x,y,z)}$:

Computation of $S_\prec(f,g)$ for $\prec = \prec_{grevlex(z,x,y)}$:

Computation of $S_\prec(f,g)$ for $\prec = \prec_{grlex(y,x,z)}$:

Exercise 2 Let f and g be two non-zero polynomials in $\Bbbk[X_1, \dots, X_n]$, and let $\prec$ be a monomial order. Let $\gamma \in \mathbb{N}^n$ be such that $X^\gamma = \text{LCM}(\text{LM}_\prec(f), \text{LM}_\prec(g))$.

Question 1: Show that $\text{LM}_\prec(S_\prec(f,g)) \prec X^\gamma$ (**!!** $\prec$ is a <u>strict</u> inequality, not <u>large</u> like $\preccurlyeq$, *i.e* we have $\alpha \preccurlyeq \alpha$ but $\alpha \not\prec \alpha$).

<u>Answer:</u>

Question 2: Prove that if $X^\alpha \prec X^\beta$, then $X^\beta \nmid X^\alpha$. (<u>Advice:</u> the properties of a monomial order can be useful → Lect. IV, Slide 4).

<u>Answer:</u>

Question 3: Deduce that <u>both</u> monomials $\text{LM}(f)$ <u>and</u> $\text{LM}(g)$ <u>can not</u> divide $\text{LM}(S(f,g))$.

<u>Answer:</u>

Exercise 3 Given $F = \{f_1, \dots, f_s\} \subset \Bbbk[X_1, \dots, X_n]$, a monomial order $\prec$ and $f \in \Bbbk[X_1, \dots, X_n]$, we know from the Property (c) of the division algorithm (Lect. III, Slide 18) that we have:

$$\exists \sigma \in \mathfrak{S}_s \text{ such that } \text{NF}(f, [f_{\sigma(1)}, \dots, f_{\sigma(s)}]) = 0 \;\Rightarrow\; f \to_F 0,$$

but $\Leftarrow$ is <u>not true</u> in general. Consider the example:

$$f_1 = x^2y^3 + 2xy^2 - 3x^2y + y^3 \qquad f_2 = x^3 + 3xy.$$

Given $a_1 = x^2 + 3x + y^2 - 1$ and $a_2 = -y^3 + 2y^3x + xy - 1$, let $f = a_1 f_1 + a_2 f_2$:

$f = -x^3 - 3xy + 3x^2y - 9x^3y - 2x^4y - 2xy^2 + 9x^2y^2 + 2x^3y^2 - y^3 + 3xy^3 - 3x^2y^3 + 2x^3y^3 + 3x^4y^3 - xy^4 + 6x^2y^4 + y^5 + x^2y^5$.

Question 1: Given $\prec = grlex(x,y)$, show that $f \to_{\{f_1,f_2\}} 0$.

Answer: *(Advice: no computations are necessary! Only the definition of f and what means "$f \to 0$" are useful)*

Question 2: However show that $\text{NF}_{\prec}(f, [f_1, f_2]) \neq 0$ and $\text{NF}_{\prec}(f, [f_2, f_1]) \neq 0$ (*i.e.* f, f_1, f_2 does not verify Property ($\star$) for $\prec$).
The division is quite complicated, so you can use Mathematica (it is very easy to use with the documentation. Check the function "PolynomialReduce". See the documentation).
Answer: (write only the remainders that you found with Mathematica...)

Exercise 4 Is the system $F = \{f_1, f_2\}$ a Gröbner basis for the ideal $I = \langle f_1, f_2 \rangle$ with respect to $\prec_{grlex(x,y)}$?

$$f_1 = -x + xy \qquad\qquad f_2 = x + x^2$$

We want to apply the Buchberger algorithm, and check that all *necessary* pairs reduce to 0.
Question 1: There is only *one* pair in this Exercise: $(1, 2)$. Is the first test (Proposition 2) applies for this pair ?

Answer:

Compute the S-polynomial $s := S_{\prec}(f_1, f_2)$.

Answer:

Question 2: Compute the division of s by one of the sequence $[f_1, f_2]$ or $[f_2, f_1]$.

Answer:

Conclude with Theorem 1 (of Lect. V)

Exercise 5 We want to compute a Gröbner basis of the polynomial system $F = \{f_1, f_2\} \subset \mathbb{k}[x, y]$ for the monomial order $\prec = \prec_{lex(x,y)}$.

$$f_1 = y^2 - y, \qquad f_2 = -x^2y + x^2 + 2xy - x + y.$$

We will follow the Buchberger algorithm, version 3 (Lect. V, Slide 19).

Question 1: At the beginning, the set of pair of indices B is simply $B = \{(1, 2)\}$.

Check if the tests 1 or 2 (Proposition 2 or 4) permits to say that $S(f_1, f_2) \to_F 0$ without computation.

Answer: Test 1 (Proposition 2) ?

Test 2 (Proposition 4) ?

If not, compute the S-polynomial $\tilde{f}_3 = S(f_1, f_2)$.

Check briefly if all the monomials of $\tilde{f}_3$ are in $\overline{\Delta}$ (Δ-sets corresponding to $[f_1, f_2]$), and if not compute the division of $\tilde{f}_3$ by $[f_1, f_2]$.

Let $f_3 = \text{NF}(\tilde{f}_3, [f_1, f_2])$. You should *not* find $f_3 = 0$. Hence, by Step 8 of the algorithm: $G = G \cup \{f_3\}$. And by Steps 9 and 11: $B = \{(1, 3), (2, 3)\}$

Question 2: Next, select the pair $(1, 3)$ in B. Check that neither Test 1 nor Test 2 work for this pair:

Test 1 ?

Test 2 ?

Compute $\tilde{f}_4 = S(f_1, f_3)$.

Check briefly that there is at least one monomial occurring in $\tilde{f}_4$ that is not in $\overline{\Delta} = \mathbb{N}^2 - \big(\Delta_1 \cup \Delta_2 \cup \Delta_3\big)$ ($\iff$ $\text{NF}(\tilde{f}_4, \{f_1, f_2, f_3\}) \neq \tilde{f}_4$).

Compute the division of $\tilde{f}_4$ by $[f_1, f_2, f_3]$.

Let f_4 the remainder $\textsc{nf}(\tilde{f}_4, [f_1, f_2, f_3])$. You should find $f_4 = 0$. By Step 9 and 11, we have: $B = \{(2, 3)\}$.

Question 3 Next, select the pair $(1, 3)$ in B. Check that neither Test 1 nor Test 2 work for this pair:

Test 1 ?

Test 2 ?

Compute $\tilde{f}_4 = S(f_1, f_3)$.

Check briefly that there is at least one monomial occurring in $\tilde{f}_4$ that is not in $\overline{\Delta} = \mathbb{N}^2 - (\Delta_1 \cup \Delta_2 \cup \Delta_3)$ ($\iff \textsc{nf}(\tilde{f}_4, \{f_1, f_2, f_3\}) \neq \tilde{f}_4$).

Compute the division of $\tilde{f}_4$ by $[f_1, f_2, f_3]$.

Let f_4 the remainder $\textsc{nf}(\tilde{f}_4, [f_1, f_2, f_3])$. You should not find $f_4 = 0$. By Step 8, we have $G = G \cup \{f_4\}$, and by Step 9 and 11, we have: $B = B - \{(2, 3)\} = \{(1, 4), (2, 4), (3, 4)\}$.

Question 4 Consider next the pair $(1, 4)$ in B. Does Test 1 or Test 2 apply for (f_1, f_4) ?

<u>Answer:</u>

Actually, it is true. So by Step 10 $B = B - \{(1, 4)\} = \{(2, 4), (3, 4)\}$.

Question 5 For the pair $(2, 4)$, check if Test 1 works.

Test 1 ?

Write all the pairs that are *not* in B, and try to see if Test 2 works.

Pairs:
Test 2 works ?

Compute the S-polynomial $\tilde{f_5} = S(f_2, f_4)$

Answer:

Compute the division of $\tilde{f_5}$ by $[f_1, f_2, f_3, f_4]$ (it is not difficult).

You should find $\text{NF}(\tilde{f_5}, [f_1, f_2, f_3, f_4]) = 0$, so by Step 10, $B = B - \{(2,4)\} = \{(3,4)\}$.

Question 6 Last, consider the pair $(3,4)$. Show that Test 1 does not work but Test 2 works. Hence it comes $B = \emptyset$ and $\{f_1, f_2, f_3, f_4\}$ is a Gröbner basis of $\langle F \rangle$ for $lex(x, y)$.

Answer:

Exercise 6 We consider a sequence of polynomials $f_1, \dots, f_s \subset \Bbbk[X_1, \dots, X_n]$, and a monomial order $\prec$.

Let $f \in \Bbbk[X_1, \dots, X_n]$, and $f = a_1 f_1 + \dots + a_s f_s + r$ the division equality.

Question 1 One property of the division, is: $a_i \neq 0 \Rightarrow \text{LM}_\prec(a_i f_i) \preccurlyeq \text{LM}_\prec(f)$ (Lect. III, Slide 18 Property (c)).

Let $\mathcal{I}(f) = \{i \mid \text{LM}_\prec(a_i f_i) = \text{LM}_\prec(f)\}$. Show that $\mathcal{I}(f) \neq \emptyset$.

Answer:

Question 2 Then show that $\text{LT}_\prec(f) = \sum_{i \in \mathcal{I}(f)} \text{LT}(a_i f_i)$.

Answer: