On elliptic curves with everywhere good reduction over certain number fields
Shun’ichi Yokoyama (Kyushu University)
draft / revised on August 8th, 2011

Abstract We prove the existence and non-existence of elliptic curves having good reduction everywhere over some real quadratic fields $K_m = \mathbb{Q}(\sqrt{m})$ for $m \leq 200$. We also give an application for the cubic field case.

1 Introduction

Throughout this paper (except Section 4), let $K_m$ be the real quadratic field $\mathbb{Q}(\sqrt{m})$ where $m$ is a square-free positive integer with $m \leq 200$ and $\mathcal{O}_{K_m}$ the ring of integers of $K_m$. We already know the following results concerning elliptic curves with everywhere good reduction over real quadratic fields ([3], [5], [6], [7], [8], [9], [13], [14], [15], [16], [18], [24], [25] and the author’s paper [26]):

Theorem 1.1. 1. There are no elliptic curves with everywhere good reduction over $K_m$ if $m = 2, 3, 5, 10, 11, 13, 15, 17, 19, 21, 23, 30, 31, 34, 35, 39, 42, 43, 46, 47, 53, 55, 57, 58, 59, 61, 66, 69, 70, 73, 74, 78, 82, 83, 85, 89, 93, 94, 95, 101, 113, 129, 141, 149, 167, 173, 177, 181, 191 and 197.

2. The elliptic curves with everywhere good reduction over $K_m$ are determined completely for $m = 6, 7, 14, 22, 29, 33, 37, 38, 41, 65, 77, 133$ and 157.

3. If $m = 62, 67$ and 71, there are no elliptic curves with everywhere good reduction over $K_m$ which have cubic discriminant (cf. [26]).

4. There are elliptic curves with everywhere good reduction over $K_m$ if $m = 26, 79, 109$ and 161 (cf. [5], [16], [25] and Cremona’s table [4]).

In this paper, we prove the existence and non-existence of elliptic curves with everywhere good reduction over certain real quadratic fields not appearing in Theorem 1.1 (except $m = 161$). Here is the main theorem:

Theorem 1.2. 1. There are no elliptic curves with everywhere good reduction over $K_m$ if $m = 103$ and 137.

2. There are admissible curves over $K_m$ if $m = 118, 134, 161$ and 166.

3. If $m = 139$ and 151, there are no admissible curves and elliptic curves with everywhere good reduction over $K_m$ which have cubic discriminant.

4. There are no admissible curves over $K_m$ if $m = 107, 127, 131, 163, 179$ and 199.
5. There is an elliptic curve $E$ with everywhere good reduction and not admissible over $K_m$ if $m = 158$ and 161:

(a) For $m = 158$, $E$ is given by $y^2 + xy + \sqrt{158}y = x^3 - x^2 + Ax + B$ where

$$A = -361817559192191668851 - 28784659475803145415\sqrt{158}$$
$$B = 3691288333191863812738417681108 + 293663132146367649175848062813\sqrt{158}$$

(b) For $m = 161$, $E$ is given by $y^2 + xy + y = x^3 - x^2 + Cx + D$ where

$$C = -3680 + 290\sqrt{161}$$
$$D = -148482 + 11702\sqrt{161}$$

In addition, there are no elliptic curves having good reduction everywhere and no $K_m$-rational point of order 2 (= not admissible) except the curve $y^2 + xy + y = x^3 - x^2 + Cx + D$ up to isogenies.

Remark 1.3. The case $m = 161$ appears in Theorem 1.1 already. However, an explicit form of the elliptic curve was not known before.

Acknowledgment It is the author’s pleasure to thank Takaaki Kagawa for his making various comments including the references to his pioneer works. The author would like to thank John Cremona and Yuichiro Taguchi who gave him some useful advice and information.

2 Strategy

In this section, we introduce the strategy to prove our results. Our strategy for the proof is close to that of T. Kagawa [9]. However, we use different kinds of computer softwares and computational techniques.

First of all, Comalada [2] determines all admissible curves (= elliptic curves having good reduction everywhere and a $K_m$-rational point of order 2) defined over $K_m$ with $m \leq 100$. [2] also gives some criteria to find admissible curves over $K_m$ for an arbitrary $m$.

Definition 2.1. An elliptic curve defined over $K_m$ is called $g$-admissible if it is admissible and has a global minimal model.

Proposition 2.2 ([2], Thm. 1). The following three conditions are equivalent:

1. There exists a $g$-admissible elliptic curve defined over $K_m$.

2. Either of the following equations has a solution in integers $u, v \in \mathcal{O}_K^\times$,

$$X, Y \in \mathcal{O}_K$$

2
(a) \( u + 64v = X^2, X \text{ a square (mod 4)}, \)
(b) \( u + v = X^2, u \equiv v \equiv 1 + 2\sqrt{m} \text{ (mod 4) and } m \equiv 2 \text{ (mod 4)}, \)
(c) \( 4Y + uY^2 = X^2, \ 2 \mid Y, \ N(Y) = \pm 16, \ N(X) \equiv -2 \text{ (mod 8) and } m \equiv 1 \text{ (mod 8)}, \) where the symbol \( N \) denotes the norm.

3. \( m = 1023 \) or either of these sets of diophantine equations has a solution:
(a) \( x^2 - 4my^2 = -7, \ 7 \mid m, \)
(b) \( x^2 - 4my^2 = 65, \ 65 \mid m, \)
(c) \( x^2 - my^2 = -2, \ m \equiv -2 \text{ (mod 8)}, \)
(d) \( x^2 - my^2 = -8 \) and \( r^4 - ms^2 = \pm 256, \ r \text{ is odd, } m \equiv 1 \text{ (mod 8)}, \)
(e) \( r^4 - ms^2 = -16384 \) and \( t^2 - mw^2 = 8r, \ r \equiv 3 \text{ (mod 4), } (t, r) = 1, \ 128w \equiv st \text{ (mod } r), \ m \equiv 1 \text{ (mod 8)}. \)

Thus we can find some admissible curves appearing Theorem 1.2 using Conalada’s method.

Next we assume that the class number of \( K_m \) is 1 and every elliptic curve \( E \) with everywhere good reduction over \( K_m \) has no \( K_m \)-rational point of order 2 (= not admissible). First we use the following result:

**Proposition 2.3** (Setzer [20]). Let \( E \) be an elliptic curve over \( K_m \). If the class number of \( K_m \) is prime to 6 then \( E \) has a global minimal model.

Let \( E \) be an elliptic curve with everywhere good reduction over \( K_m \). By Proposition 2.3, \( E \) has a global minimal model

\[
E: y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6
\]

with coefficients \( a_i \in O_{K_m} \) (\( i = 1, 2, 3, 4, 6 \)). The discriminant of \( E \) (denoted by \( \Delta(E) \)) is

\[
\Delta(E) = \frac{c_4^3 - c_6^2}{1728}
\]

where \( c_4, c_6 \in O_{K_m} \) are, as in [23] (Chapter III, p.42), written as polynomials in the \( a_i \)'s with integer coefficients. Moreover, the following conditions are equivalent (cf. [23], Chapter VII, Prop. 5.1):

- \( E \) has everywhere good reduction over \( K_m \),
- \( \Delta(E) \in O_{K_m}^\times. \)

In our case, all elements of \( O_{K_m}^\times \) are written in the form \( \pm \varepsilon^n \) where \( \varepsilon \) is a fundamental unit of \( K_m \) (let us fix \( \varepsilon \) for each \( m \)). Thus to determine the elliptic curves with everywhere good reduction over \( K_m \), we shall compute the sets

\[
E_n^\pm (O_{K_m}) = \{(x, y) \in O_{K_m} \times O_{K_m} \mid y^2 = x^3 \pm 1728 \varepsilon^n \}, \ 0 \leq n < 12.
\]
However, the set of coefficients \((a_1, a_2, a_3, a_4, a_6) \in \mathcal{O}_{K_m}^5\), which gives rise to \((c_4, c_6) \in \mathcal{O}_{K_m}^{\pm 2}\), does not necessarily exist. Therefore, we check whether the curve

\[
E_C : y^2 = x^3 - 27c_4x - 54c_6,
\]

which is isomorphic to \(E\) over \(K_m\), has trivial conductor for each \((c_4, c_6) \in E_n(\mathcal{O}_{K_m})\).

Actually, it is very hard to compute all \(E_n(\mathcal{O}_{K_m})\) because of the limitation of efficiency of equipments. To reduce the amount of computation, we show that some values of \(n\) are irrelevant by using Kagawa’s results. In [9], Kagawa shows a criterion whether the discriminant of an elliptic curve with everywhere good reduction over \(K_m\) is a cube in \(K_m\):

**Lemma 2.4** ([9], Prop. 1). If the following five conditions hold, then the discriminant of every elliptic curve with everywhere good reduction over \(K_m\) is a cube in \(K_m\):

1. The class number of \(K_m\) is prime to 6;
2. \(K_m/\mathbb{Q}\) is unramified at 3;
3. The class number of \(K_m(\sqrt{-3})\) is prime to 3;
4. The class number of \(K_m(\sqrt{3})\) is odd;
5. For some prime ideal \(p\) of \(K_m\) dividing 3, the congruence \(X^3 \equiv \varepsilon (\mod p^2)\) does not have a solution in \(\mathcal{O}_{K_m}\).

Using the criterion, Kagawa shows the following:

**Lemma 2.5** ([11]). If \(m = 107, 127, 161, 166\) or 193, every elliptic curve with everywhere good reduction over \(K_m\) has a global minimal model whose discriminant is a cube in \(K_m\).

Therefore, we have \(\Delta(E) = \pm 3^n\) for some \(n \in \mathbb{Z}\).

By applying the next lemma, we can further discard some cases:

**Lemma 2.6** ([9], Prop. 4). Let \(E\) be an elliptic curve defined over \(K_m\). If \(E\) has good reduction outside 2 and has no \(K_m\)-rational point of order 2, then \(K_m(E[2])/K_m(\sqrt{\Delta(E)})\) is a cyclic cubic extension unramified outside 2. In particular, the ray class number of \(K_m(\sqrt{\Delta(E)})\) modulo \(\prod_{p|2}p\) is a multiple of 3.

Note that \(K_m(\sqrt{\Delta(E)})\) is either \(K_m, K_m(\sqrt{-1})\) or \(K_m(\sqrt{\pm 3})\). Thus we compute the ray class number of \(K_m(\sqrt{\Delta(E)})\) modulo \(\prod_{p|2}p\). The following computations are carried out by using Pari/GP [17] (Same type results were obtained in [10] by using KASH [12]). The bold-faced numbers in this table are the ones divisible by 3.
To compute $E_n(K_m)$, we first compute the Mordell-Weil group $E_n(K_m)$. It is decomposed into a direct-sum of $E_n(K_m)_{\text{tors}}$ (torsion part) and $E_n(K_m)_{\text{free}}$ (free part, which is not canonical). The torsion part can be determined by observing reduction at good primes and decomposition of division polynomials. On the other hand, the free part can be computed by applying two-descent and infinite descent (the process of decompression from $E_n(K_m) = 2E_n(K_m)$ to $E_n(K_m)$). We used Denis Simon’s two-descent program (cf. [21]) on Pari-GP [17]. To compute some related data efficiently, we executed the Pari-GP program on Sage [19] as a built-in software.

To compute the subset $E_n(K_m)$ of integral points in $E_n(K_m)$, we use the method of elliptic logarithm to compute the linear form:

$$P = \sum_{i=1}^{r} m_i P_i + nT \in E_n(K_m) \quad (m_1, \ldots, m_r, n \in \mathbb{Z})$$

where $P_i$’s and $T$ are generators of the free part and the torsion part. Moreover, the maximum of the absolute values of the coefficients of the linear form

$$M := \max \{|m_1|, \ldots, |m_r|, |n|\}$$

can be bounded using the LLL-algorithm (by Lenstra-Lenstra-Lovász, cf. [22]). Finally, we compute that the elliptic curve (1) has trivial conductor.
3.1 Admissible curves

First we prove the existence / non-existence of $g$-admissible curves:

**Proposition 3.1.**

1. There are no $g$-admissible curves over $K_m$ if $m = 103, 107, 127, 131, 137, 139, 151, 158, 163, 179$ and $199$.

2. There are $g$-admissible curves over $K_m$ if $m = 118, 134, 161$ and $166$.

**Proof.** 1. The third equivalent condition (a)–(e) of Proposition 2.2 does not be satisfied except for $m = 158$. In the exceptional case, there exists a $g$-admissible curve over $K_m$ if and only if $m = 2q, q \equiv 3 \pmod{4}, x^2 - my^2 = -2$ is solvable.

We can compute that $q = 79, q \equiv 3 \pmod{4}$ and the equation $x^2 - 158y^2 \equiv -2 \pmod{q}$ has no solutions. Thus we conclude that also the equation $x^2 - my^2 = -2$ is non-solvable in $\mathcal{O}_{K_m}$. Thus the non-existence of $g$-admissible curves follows.

2. For $m = 118, 134$ and $166$, we need to check whether

$$x^2 - my^2 = -2$$

has a solution in $\mathcal{O}_{K_m}$. As a result, we get the following results:

- (Case $m = 118$) $554^2 - 118 \cdot 51^2 = -2$,
- (Case $m = 134$) $382^2 - 134 \cdot 33^2 = -2$,
- (Case $m = 166$) $41242^2 - 166 \cdot 3201^2 = -2$.

For $m = 161$, we need to check whether

$$x^2 - 4my^2 = -7$$

has a solution in $\mathcal{O}_{K_m}$, and we can get $203^2 - 4 \cdot 161 \cdot 8^2 = -7$.

**Remark 3.2.** Using Lemma 2.6 and Table 1, it follows that there are no elliptic curves with everywhere good reduction over $K_m$ if $m = 103$ and $137$.

**Remark 3.3.** In fact, [2] proved that the number of $g$-admissible elliptic curves over $K_m$ (up to isomorphism) for $m = 118, 134, 166$ is $2$ if $2$ is solvable and $m \neq 6$. Thus we conclude that the number of admissible elliptic curves over $K_m$ for $m = 118, 134, 166$ is greater than or equal to $2$. Note that it is not true in general that all admissible curves defined over $K_m$ are $g$-admissible. However, assume the class number of $K_m$ is odd, it is true except some cases (see [2]).
3.2 For \( m = 139 \) and 151

In this case, it is enough to determine \( E_0^+ (\mathcal{O}_K_{139}) \) and \( E_3^- (\mathcal{O}_K_{151}) \). Here is the result of computing Mordell-Weil groups and sets of integral points:

**Proposition 3.4.** A basis of \( E_0^+ (K_m) \) and the set of integral points \( E_n^+ (\mathcal{O}_K_m) \) are as follows:

1. (Case \( m = 139 \)) \( E_0^+ (K_{139}) \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) with a basis \( \{ T_{139}, P_{139} \} \) where \( T_{139} = (-12, 0) \) is 2-torsion and

\[
P_{139} = \left( -\frac{21}{4}, -\frac{27}{8} \sqrt{139} \right)
\]

is a generator of the free-part. The set of integral points is

\[
E_0^+ (\mathcal{O}_{K_{139}}) = \{ O, T_{139}, T_{139} \pm P_{139} \}.
\]

Moreover, there are no pairs \((c_4, c_6) \in E_n^+ (\mathcal{O}_m)\) for which the elliptic curve (1) has trivial conductor.

2. (Case \( m = 151 \)) \( E_3^- (K_{151}) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) with a basis \( \{ T_{151} \} \) where \( T_{151} = (12\varepsilon, 0) \) (\( \varepsilon = -1728148040 + 140634693\sqrt{151} \)) is 2-torsion. The set of integral points is

\[
E_3^- (\mathcal{O}_{K_{151}}) = \{ O, T_{151} \}.
\]

3.3 For \( m = 161 \)

In this case, we can find \((c_4, c_6)\) (that gives the elliptic curve with everywhere good reduction appearing in Theorem 1.2) from \( E_3^- (\mathcal{O}_{K_{161}}) \).

**Proposition 3.5.** \( E_3^- (K_{161}) \) is isomorphic to \( \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \) with a basis \( \{ T_{161}, P_{161A}, P_{161B} \} \) where \( T_{161} = (12\varepsilon, 0) \) (\( \varepsilon = -11775 + 928\sqrt{161} \)) is 2-torsion and

\[
P_{161A} = \left( 47100 - 3712\sqrt{161}, -76493648 + 6028544\sqrt{161} \right)
\]

and

\[
P_{161B} = \left( \frac{168519875}{2028} - \frac{9960920}{1521} \sqrt{161}, -\frac{4887926943802}{59319} + \frac{3081780871477}{474552} \sqrt{161} \right)
\]

are generators of the free-part. The set of integral points is

\[
E_3^- (\mathcal{O}_{K_{161}}) = \{ O, T_{161}, \pm P_{161A}, T_{161} \pm P_{161A} \}.
\]

(double sign in the same order).

Finally, we can get \((c_4, c_6) = T_{161} + P_{161A}\) and find the curve explained in Theorem 1.2.
3.4 For \( m = 158 \)

In this case, the class number of \( K_m \) is 2 so our strategy cannot apply. However, we can find one elliptic curve with everywhere good reduction over \( K_m \) with computing \( E_3(\mathcal{O}_{K_{158}}) \).

**Proposition 3.6.** \( E_3(K_{158}) \) is isomorphic to \( \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \) with a basis \( \{ T_{158}, P_{158A}, P_{158B} \} \) where \( T_{158} = (12\varepsilon, 0) \) \( ( \varepsilon = 7743 + 616\sqrt{158} ) \) is 2-torsion and

\[
P_{158A} = \left( 5559122 + 442260\sqrt{158}, 18536310324 + 1474669670\sqrt{158} \right)
\]

and

\[
P_{158B} = \left( 154860 + 12320\sqrt{158}, 76310080 + 6070904\sqrt{158} \right)
\]

are generators of the free-part. The set of integral points is

\[
E_3(\mathcal{O}_{K_{158}}) = \{ O, T_{158}, \pm P_{158A}, \pm P_{158B}, \pm P_{158A} \pm P_{158B},
\]

\[
\pm P_{158A} \mp 2P_{158B}, T_{158} \pm P_{158A} \mp P_{158B}, T_{158} \pm P_{158B} \}
\]

(double sign in the same order).

Finally, we can get \( (c_4, c_6) = T_{158} - P_{158A} + P_{158B} \) and find the curve explained in Theorem 1.2.

4 Application for the cubic field case

In this section, we give an application for the (real) cubic field case using our approach. The only known result is given by Bertolini-Canuto [1]:

**Theorem 4.1** (Bertolini-Canuto, 1988). Let \( K \) be the field \( \mathbb{Q}(\alpha) \) where \( \alpha \) is the real cube root of 2. Then there are no elliptic curves over \( K \) with good reduction everywhere.

[1] prove at first that all the elliptic curves defined over \( \mathbb{Q}(\alpha) \) with good reduction everywhere have a point of order 2 rational, i.e. admissible over \( \mathbb{Q}(\alpha) \). After that, [1] prove the non-existence of admissible curves over \( \mathbb{Q}(\alpha) \).

Structures of cubic number fields \( K_m \) are similar to that of real quadratic fields. Let \( L_m \) be the cubic field \( \mathbb{Q}(\sqrt[3]{m}) \) where \( m \) is not cubic, positive integer with \( m \leq 100 \) and \( \mathcal{O}_{L_m} \) the ring of integers of \( L_m \). We note that in Proposition 2.3 and Lemma 2.6, the base field \( K_m \) can be replaced to \( L_m \). Thus we compute the ray class number of \( L_m(\sqrt{\Delta(E)}) \) modulo \( \prod_{p|\Delta} p \) for \( L_m \) whose class number is prime to 6. In fact, for \( m \leq 100 \), if the class number of \( L_m \) is prime to 6, the class number is 1.

We also note that for all \( m \) listed Table 2, the unit group \( \mathcal{O}_{L_m}^\times \) is isomorphic to \( \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), i.e. \( \langle -1, \varepsilon \rangle \) where \( \varepsilon \) is a fundamental unit of \( L_m \) (let us fix \( \varepsilon \) for each \( m \)).
<table>
<thead>
<tr>
<th>$m$</th>
<th>$L_m$</th>
<th>$L_m(\sqrt{-1})$</th>
<th>$L_m(\sqrt{2})$</th>
<th>$L_m(\sqrt{-2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>23</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>29</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>33</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>36</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>41</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>44</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>45</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>46</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>53</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>55</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>59</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>69</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>71</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>75</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>82</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>87</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>99</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2. Ray class number of $L_m(\sqrt{\Delta(E)})$ modulo $\prod p|2$ $p$

Using the table above, we conclude immediately as follows:

**Proposition 4.2.** If $m = 3, 4, 5, 6, 9, 10, 12, 17, 18, 25, 29, 36$ and $100$, every elliptic curve $E$ defined over $L_m$ with everywhere good reduction is admissible.

# We will add data of computing Mordell-Weil groups (in preparation).
A Unsolved cases

We give the list of undetermined Mordell-Weil groups $E_n^\pm(K_m)$ for $m \leq 200$ and the class number is 1.

Case $m = 62$ It is enough to determine $E_1^-, E_3^-$ and $E_5^-.$

<table>
<thead>
<tr>
<th>$E_n^-$</th>
<th>rank</th>
<th>generators (free part)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1^-$</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$E_3^-$</td>
<td>1</td>
<td>$G_{1A}$</td>
</tr>
<tr>
<td>$E_5^-$</td>
<td>1</td>
<td>--</td>
</tr>
</tbody>
</table>

fundamental unit: $\varepsilon = -63 + 8\sqrt{62}$

where $G_{1A} = \left(\frac{30492}{25} - \frac{3872}{25}\sqrt{62}, \frac{837796}{125} + 8512\sqrt{62}\right)$.

Case $m = 67$ It is enough to determine $E_0^+, E_2^+$ and $E_4^+$.

<table>
<thead>
<tr>
<th>$E_n^+$</th>
<th>rank</th>
<th>generators (free part)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0^+$</td>
<td>1</td>
<td>$G_{2A}$</td>
</tr>
<tr>
<td>$E_2^+$</td>
<td>1</td>
<td>--</td>
</tr>
<tr>
<td>$E_4^+$</td>
<td>1</td>
<td>--</td>
</tr>
</tbody>
</table>

fundamental unit: $\varepsilon = -48842 + 5967\sqrt{67}$

where $G_{2A} = \left(-\frac{584}{49}, \frac{248}{49}\sqrt{67}\right)$.

Case $m = 71$ It is enough to determine $E_1^-, E_3^-$ and $E_5^-.$

<table>
<thead>
<tr>
<th>$E_n^-$</th>
<th>rank</th>
<th>generators (free part)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1^-$</td>
<td>1 $\leq r \leq 3$</td>
<td>$G_{1A}, ??$</td>
</tr>
<tr>
<td>$E_3^-$</td>
<td>2</td>
<td>$G_{3A}, G_{3B}, G_{3C}$</td>
</tr>
<tr>
<td>$E_5^-$</td>
<td>1 $\leq r \leq 3$</td>
<td>??</td>
</tr>
</tbody>
</table>

fundamental unit: $\varepsilon = 3480 + 413\sqrt{71}$

where

\[
G_{1A} = \left(\frac{15625056}{49} + \frac{1782764}{49}\sqrt{71}, -\frac{82351180712}{443} - \frac{9773265400}{443}\sqrt{71}\right),
\]

\[
G_{3A} = \left(\frac{156462848}{3025} + \frac{185192688}{3025}\sqrt{71}, -\frac{87152513416872}{166375} - \frac{10343100438152}{166375}\sqrt{71}\right),
\]

\[
G_{3B} = \left(\frac{156462848}{3025} + \frac{185192688}{3025}\sqrt{71}, -\frac{87152513416872}{166375} - \frac{10343100438152}{166375}\sqrt{71}\right),
\]

Case $m = 107$ It is enough to determine $E_0^+$ and $E_3^+$.

<table>
<thead>
<tr>
<th>$E_n^+$</th>
<th>rank</th>
<th>generators (free part)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0^+$</td>
<td>1</td>
<td>$G_{4A}$</td>
</tr>
<tr>
<td>$E_3^+$</td>
<td>0 $\leq r \leq 2$</td>
<td>$G_{4B}, ??$</td>
</tr>
</tbody>
</table>

fundamental unit: $\varepsilon = 962 + 93\sqrt{107}$
where

\[ G_{4A} = \left( \frac{19415435}{53824} \frac{827055739}{1245748} \sqrt{107}, \right), \]
\[ G_{4B} = \left( \frac{13990698150670419032}{982587075660025} + \frac{1265524873193709948}{982587075660025} \right) \frac{\sqrt{107}}{107}, \]
\[ - \frac{860549180004386357832502428984}{3090011205559190606125} - \frac{831924756766298779020529400}{3090011205559190606125} \frac{\sqrt{107}}{107}. \]

**Case** \( m = 109 \)  
It is enough to determine \( E_3^{-} \).

<table>
<thead>
<tr>
<th>( E_3^{-} )</th>
<th>rank</th>
<th>generators (free part)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_3^{-} )</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>( E_3^{-} )</td>
<td>1</td>
<td>??</td>
</tr>
</tbody>
</table>

fundamental unit: \( \varepsilon = \frac{261}{7} + 2 \frac{\sqrt{109}}{7} \)

where

\[ G_{5A} = \left( \frac{10154}{81} + \frac{970}{81} \sqrt{109}, - \frac{4210060}{729} - \frac{403268}{729} \sqrt{109} \right), \]
\[ G_{5B} = \left( \frac{916346}{81} + \frac{87770}{81} \sqrt{109}, - \frac{1613792380}{729} - \frac{154573276}{729} \sqrt{109} \right). \]

**Case** \( m = 127 \)  
It is enough to determine \( E_3^{-} \).

<table>
<thead>
<tr>
<th>( E_3^{-} )</th>
<th>rank</th>
<th>generators (free part)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_3^{-} )</td>
<td>0 ( \leq r \leq 2 )</td>
<td>??</td>
</tr>
</tbody>
</table>

fundamental unit: \( \varepsilon = -4730624 + 419775 \sqrt{127} \)

**Case** \( m = 131 \)  
It is enough to determine \( E_0^+ \), \( E_2^+ \), \( E_4^+ \), \( E_1^- \), \( E_3^- \) and \( E_5^- \).

<table>
<thead>
<tr>
<th>( E_5^- )</th>
<th>rank</th>
<th>generators (free part)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_0^+ )</td>
<td>1</td>
<td>( G_{7A} )</td>
</tr>
<tr>
<td>( E_2^+ )</td>
<td>1</td>
<td>??</td>
</tr>
<tr>
<td>( E_4^+ )</td>
<td>1</td>
<td>??</td>
</tr>
<tr>
<td>( E_1^- )</td>
<td>0 ( \leq r \leq 2 )</td>
<td>??</td>
</tr>
<tr>
<td>( E_3^- )</td>
<td>2</td>
<td>( G_{7B} ) ??</td>
</tr>
<tr>
<td>( E_5^- )</td>
<td>0 ( \leq r \leq 2 )</td>
<td>??</td>
</tr>
</tbody>
</table>

fundamental unit: \( \varepsilon = -10610 + 927 \sqrt{131} \)

where

\[ G_{7A} = \left( \frac{-43296}{7225} + \frac{2208712}{614125} \right) \sqrt{131}, \]
\[ G_{7B} = \left( \frac{1587256}{5} - \frac{693396}{25} \right) \sqrt{131}, - \frac{32622430104}{125} - \frac{2850234952}{125} \sqrt{131}. \]

**Case** \( m = 139 \)  
It is enough to determine \( E_0^+ \), \( E_2^+ \) and \( E_4^+ \).
$E_n$  rank  generators (free part)
\[\begin{array}{|c|c|}
\hline
E_0  & 1  & G_{8A} \\
E_2  & 1  & ?? \\
E_4  & 1  & ?? \\
\hline
\end{array}\]

fundamental unit: $\varepsilon = -77563250 + 6578829\sqrt{139}$

where $G_{8A} = \left(-\frac{21}{4}, -\frac{27}{8}\sqrt{139}\right)$.

**Case $m = 151$**  It is enough to determine $E_1^-$, $E_3^-$ and $E_5^-$.  

$E_n$  rank  generators (free part)
\[\begin{array}{|c|c|}
\hline
E_0  & 0 \leq r \leq 2  & ?? \\
E_2  & 0 \leq r \leq 2  & ?? \\
E_4  & 2  & G_{9A}, ?? \\
\hline
\end{array}\]

fundamental unit: $\varepsilon = -1728148040 + 140634693\sqrt{151}$

where $G_{9A} = (-3346088623246672 + 272300830362362\sqrt{151}$, $14594621900373131762079562 - 1187693486265246101920898\sqrt{151})$.

**Case $m = 163$**  It is enough to determine $E_0^+$, $E_1^+$, $E_2^+$, $E_3^+$, $E_4^+$ and $E_5^+$.

$E_n$  rank  generators (free part)
\[\begin{array}{|c|c|}
\hline
E_0  & 0 \leq r \leq 2  & ?? \\
E_2  & 0 \leq r \leq 2  & ?? \\
E_4  & 0 \leq r \leq 2  & ?? \\
E_5  & 0 \leq r \leq 3  & ?? \\
\hline
\end{array}\]

fundamental unit: $\varepsilon = -64080026 + 5019135\sqrt{163}$

where  

\[G_{10A} = (640320, 40133016\sqrt{163}), \]
\[G_{10B} = (1281600520 - 100382700\sqrt{163}, -57451588558840 + 4499955710712\sqrt{163}).\]

**Case $m = 179$**  It is enough to determine $E_0^+$, $E_1^+$, $E_2^+$, $E_3^+$ and $E_4^+$.  

$E_n$  rank  generators (free part)
\[\begin{array}{|c|c|}
\hline
E_0  & 0 \leq r \leq 2  & ?? \\
E_2  & 0 \leq r \leq 2  & ?? \\
E_4  & 0 \leq r \leq 2  & ?? \\
E_5  & 0 \leq r \leq 2  & ?? \\
\hline
\end{array}\]

fundamental unit: $\varepsilon = 4190210 + 313191\sqrt{179}$
where \( G_{11\mathcal{A}} = ( -\frac{250557111908}{113819191641}, \frac{11893889216317192}{383929353015581} \sqrt{179} ). \)

**Case \( m = 193 \)** It is enough to determine \( E_3^+ \).

<table>
<thead>
<tr>
<th>( E_{11}^+ )</th>
<th>rank</th>
<th>generators (free part)</th>
<th>( G_{12\mathcal{A}} )</th>
<th>??</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_3^+ )</td>
<td>2</td>
<td>( G_{12\mathcal{A}} )</td>
<td>??</td>
<td></td>
</tr>
</tbody>
</table>

fundamental unit: \( \varepsilon = 1764132 + 126985\sqrt{193} \)

where \( G_{12\mathcal{A}} = \left( \frac{7409865}{8} + \frac{5392062}{8} \sqrt{193}, \frac{-2297140821745}{16} - \frac{-165366210833}{16} \sqrt{193} \right). \)

**Case \( m = 199 \)** It is enough to determine \( E_0^+, E_2^+, E_4^+ \).

<table>
<thead>
<tr>
<th>( E_{13}^+ )</th>
<th>rank</th>
<th>generators (free part)</th>
<th>( G_{13\mathcal{A}} )</th>
<th>??</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_0^+ )</td>
<td>0 ( \leq r \leq 1 )</td>
<td>( G_{13\mathcal{A}} )</td>
<td>??</td>
<td></td>
</tr>
<tr>
<td>( E_2^+ )</td>
<td>0 ( \leq r \leq 1 )</td>
<td>( G_{13\mathcal{A}} )</td>
<td>??</td>
<td></td>
</tr>
</tbody>
</table>

fundamental unit: \( \varepsilon = -16266196520 + 1153080099\sqrt{199} \)

where \( G_{13\mathcal{A}} = \left( \frac{527238916}{6447025}, \frac{85956595248}{16484028625} \sqrt{199} \right). \)

**References**


Shun’ichi Yokoyama  
Graduate School of Mathematics, Kyushu University  
744 Motooka, Nishi-ku, Fukuoka, 819-0395, Japan  
E-mail address:  
s-yokoyama@math.kyushu-u.ac.jp